1 Green's functions and retarded potentials in electrodynamics

1.1 Green's functions recap

Consider an inhomogeneous differential equation of form

$$L_x u(x) = f(x), \tag{1.1}$$

where L is a general *linear* operator, which acts on coordinate x (e.g., ∂^2). If we could "invert" the operator, we could easily find solutions $u = L^{-1}f$.

With this thought as a loose motivation, we associate to L a *Green's function*, G, which is defined via:

$$L_x G(x, x') \equiv \delta(x - x'), \tag{1.2}$$

which at least kind of *looks* like an inverse. But is this useful? Well, assuming G is known, notice that if we act on $f(x)$ with $L_x G$ and integrate, we may write

$$\int \mathrm{d}x' L_x G(x, x') f(x') = \int \mathrm{d}x' \delta(x - x') f(x')$$
$$L_x \int \mathrm{d}x' G(x, x') f(x') = f(x), \tag{1.3}$$

where we moved the linear L_x operator through the integral sign on the left-hand-side, since it acts only on coordinate x (not x'), and used the Dirac delta to evaluate the integral on the right-hand-side. By comparing with Eq. (1.1), we recognise the remaining integral as $u(x)$. As such, presuming G can be found by solving (1.2), we may easily find solutions to the inhomogeneous equation:

$$u(x) = \int \mathrm{d}x' G(x, x') f(x'), \tag{1.4}$$

called the *Green's function solution* to the differential equation.

In general, the differential operator L may also admit solutions to the *homogeneous equation*

$$L u_0(x) = 0.$$

Since the operator is linear, the most general solution to Eq. (1.1) can be written as the sum $u(x) = u_0(x) + u_*(x)$, where $u_*(x)$ is a particular solution satisfying $Lu_*(x) = f(x)$. The Green's function expression in Eq. (1.4) gives such a particular solution $u_*(x)$. Therefore, the *general solution* to the inhomogeneous equation is:

$$u(x) = u_0(x) + \int \mathrm{d}x'\, G(x, x') f(x'). \tag{1.5}$$

The homogeneous component $u_0(x)$ is fixed by the boundary or initial conditions. In practical problems, we can often choose $u_0 = 0$.

1.2 Poisson equation (electrostatic)

As a concrete example, we'll consider the 3D static Poisson equation (which follows directly from Gauss' law):

$$\nabla^2 \phi(\boldsymbol{x}) = -\rho(\boldsymbol{x}). \tag{1.6}$$

The corresponding Green's function is defined:

$$\nabla^2 G \equiv \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'). \tag{1.7}$$

Noting the spherical symmetry of the Laplacian and the delta, we conclude that $G = G(r)$, where $r = |\boldsymbol{x} - \boldsymbol{x}'|$. It is therefore also convenient to write the Laplacian in spherical coordinates. Away from the origin, the delta function is zero, and so we have

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}G}{\mathrm{d}r}\right) = 0 \quad \text{for} \quad r \neq 0.$$

The simplest way to solve this is by defining $\chi = rG$, in which case the equation becomes $\chi'' = 0$. Integrating twice, we see $\chi = a + br$, or

$$G = \frac{a}{r} + b. \tag{1.8}$$

Requiring that $G \to 0$ as $r \to \infty$ allows us to set $b = 0$. We find the constant a by integrating Eq. (1.7) over a small sphere of radius R around $r = 0$ using Gauss' theorem:

$$\oint_V (\nabla^2 G)\, \mathrm{d}^3 r = \oint_S (\nabla G) \cdot \mathrm{d}\boldsymbol{S} = 1,$$

where S is the surface of the volume, $\mathrm{d}\boldsymbol{S}$ is the surface element (outwardly) normal to the surface, and we performed the integral on the right-hand-side of (1.7) by noting that the delta function integrates to 1. We have $\nabla \frac{1}{r} = -\frac{1}{r^3}\boldsymbol{r}$, so

$$-a \oint_S \frac{1}{r^2}(\hat{\boldsymbol{n}} \cdot \mathrm{d}\boldsymbol{S}) = -a\frac{1}{R^2}\oint_S (\hat{\boldsymbol{n}} \cdot \mathrm{d}\boldsymbol{S}) = -a\frac{1}{R^2}(4\pi R^2) = 1,$$

where $\hat{\boldsymbol{n}} = \boldsymbol{r}/|r|$ is parallel to the surface element $\mathrm{d}\boldsymbol{S}$, so the integral is just the surface area of the sphere S. Thus, we find $a = -1/(4\pi)$, and so the Green's function is:

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{-1}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{1.9}$$

Finally, we have the solution to the Poisson equation:

$$\phi(\boldsymbol{x}) = \frac{1}{4\pi}\int \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\mathrm{d}^3 x'. \tag{1.10}$$

It's important to note that the spherical symmetry argument that applied for the Green's function certainly does not apply (in general) for the Poisson equation.

1.2.1 Fourier transform method

We may also find the Green's function (1.7) in perhaps a more straight-forward way by using the method of Fourier transforms. We use the Fourier representation of the delta function,

$$\delta(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.11}$$

and of the Green's function:

$$G(\boldsymbol{x} - \boldsymbol{x}') = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \widetilde{G}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{x}')}. \tag{1.12}$$

Then, inserting into (1.7), and acting with the differential operator under the integral sign, we have

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \widetilde{G}(\boldsymbol{k})\, \nabla^2 e^{i\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{x}')} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{x}')}, \tag{1.13}$$

from which we can read off

$$(i\boldsymbol{k}) \cdot (i\boldsymbol{k})\, \widetilde{G}(\boldsymbol{k}) = 1. \tag{1.14}$$

Plugging back into Eq. (1.12), we find the integral expression for the Green's function:

$$\begin{aligned} G(\boldsymbol{x}-\boldsymbol{x}') &= -\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{k^2} \\ &= -2\pi\int_0^\infty \frac{k^2\mathrm{d}k}{(2\pi)^3}\int_0^\pi \mathrm{d}\theta\,\sin\theta\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{k^2} \\ &= -\frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}k\int_{-1}^1 \mathrm{d}\gamma\, e^{ik(x-x')\gamma}, \end{aligned} \tag{1.15}$$

where we took advantage of the spherical symmetry, with $k=|\boldsymbol{k}|$ and $x=|\boldsymbol{x}|$, and made change of variables $\gamma=\cos\theta$. Continuing, we have

$$\begin{aligned} G(\boldsymbol{x}-\boldsymbol{x}') &= -\frac{2}{(2\pi)^2}\int_0^\infty \frac{\sin(kr)}{kr}\,\mathrm{d}k \\ &= \frac{-1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}, \end{aligned} \tag{1.16}$$

where I introduced $r=|\boldsymbol{x}-\boldsymbol{x}'|$ for brevity. The final integral is a little tricky due to the removable singularity at $k=0$, though can be performed using complex analysis (or just with a standard table of integrals!).

1.3 Electrodynamics: retarded potentials

From Maxwell's inhomogeneous equation, we have[1]

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{1}{c}j^\mu \\ \partial^2 A^\nu - \partial^\nu\partial_\mu A^\mu &= \frac{1}{c}j^\nu. \end{aligned} \tag{1.17}$$

By imposing the Lorenz gauge choice, $\partial_\mu A^\mu=0$, this simplifies to:

$$\partial^2 A^\mu = \frac{1}{c}j^\mu, \tag{1.18}$$

which may be called the *inhomogeneous d'Alembert* equation. In terms of the perhaps more familiar scalar and vector potentials, we have[2]

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}-\nabla^2\Phi=\rho, \quad \text{and} \quad \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}-\nabla^2\boldsymbol{A}=\frac{1}{c}\boldsymbol{j}, \tag{1.19}$$

which are general (inhomogeneous) wave equations. In the absence of charges or currents ($\rho=j=0$), the solutions include the familiar electromagnetic plane waves.

We shall now consider the general case of the potentials due to time-dependent charge and current distributions. For simplicity, we will consider the scalar potential; the derivation for the vector potential follows similarly. Beginning from the inhomogeneous d'Alembert equation (1.18), and maintaining the Lorenz Gauge condition, we have

$$\partial^2\Phi(\boldsymbol{x},t)=\rho(\boldsymbol{x},t). \tag{1.20}$$

Since this is a linear equation, the general solution can be expressed as the sum of solutions due to charge distributions of separate infinitesimal pieces of space. As such, we first consider just the solution due to a small point charge $\mathrm{d}q$, located at position $\boldsymbol{x}_0$. We thus write $\rho(t,\boldsymbol{x})=\mathrm{d}q(t)\delta(\boldsymbol{x}-\boldsymbol{x}_0)$, so that

$$\partial^2\Phi=\mathrm{d}q(t)\delta(\boldsymbol{r}), \tag{1.21}$$

where we introduce the variable $\boldsymbol{r}\equiv\boldsymbol{x}-\boldsymbol{x}_0$, noting that $\boldsymbol{x}_0$ is a constant (the t-dependence has been absorbed into $\mathrm{d}q$ – I am imagining a constant region of space, with a variable amount of charge in it). We shall solve this first using the standard "textbook" method ("method of undetermined coefficients", see Griffiths for example), and then using the Fourier transform method.

1.3.1 Method of undetermined coefficients

Since we consider just a single point charge, the problem has a clear spherical symmetry: Φ is a function only of $r=|\boldsymbol{r}|$. Everywhere except at $r=0$, we have $\partial^2\Phi=0$. Writing the Laplacian in spherical coordinates:

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)=0 \quad \text{for} \quad r\neq 0,$$

and using the common trick of defining $\chi\equiv r\Phi$, we have:

$$\frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2}-\frac{\partial^2\chi}{\partial r^2}=0 \quad \text{for} \quad r\neq 0, \tag{1.22}$$

which is exactly the one-dimensional wave equation, which has periodic solutions of the form $e^{i(\omega t-kr)}$, with $k=\pm\omega/c$. In other words, solutions are of the form $\chi_\pm(t-\pm r/c)$. Since we need just one particular solution, we take the positive case, and thus write $\chi=\chi_+(t-r/c)$, or

$$\Phi=\frac{\chi_+(t-r/c)}{r}. \tag{1.23}$$

We may determine the specific form of χ_+, and thus Φ, by asserting that Φ approaches the correct solution to (1.21) as $r\to 0$. Notice that, for periodic χ solutions, the potential $\Phi\to\infty$ as $r\to 0$ from Eq. (1.23). As such, close to $r=0$, the time derivatives in Eq. (1.21) are negligible compared to the spatial derivatives, and the solution has the same form as the static case we found in Eq. (1.10):

$$\Phi(\boldsymbol{x})_{\rm static}=\frac{1}{4\pi}\int\frac{\rho(\boldsymbol{x}')}{r}.$$

Therefore, we have $\chi(t)=\frac{1}{4\pi}\mathrm{d}q(t)$, and

$$\Phi(\boldsymbol{x},t)=\frac{1}{4\pi}\frac{\mathrm{d}q(t-r/c)}{r}. \tag{1.24}$$

The extension to general charge distributions is clear: write $\mathrm{d}q=\rho\mathrm{d}V$, and integrate over the entire space:

$$\Phi_r(\boldsymbol{x},t)=\frac{1}{4\pi}\int\frac{\rho(\boldsymbol{x}',t_r)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3r', \tag{1.25}$$

where $t_r=t-|\boldsymbol{x}-\boldsymbol{x}'|/c$. The derivation for the other components of the potential follow in exactly the same way, resulting in

$$A^\mu_r(\boldsymbol{x},t)=\frac{1}{4\pi\,c}\int\frac{j^\mu(\boldsymbol{x}',t_r)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x', \tag{1.26}$$

which is known as the *retarded potential*.

[1] I am using Heaviside-Lorentz units.

[2] If unfamiliar with the covariant form of Maxwell's equations, Eq. (1.19) may be derived directly from the 3-vector form of Maxwell's two inhomogeneous equations by expressing them in terms of the potentials, and imposing the Lorenz condition.

The retarded potential, A^μ_r, at position $\boldsymbol{x}$ and time t, is thus understood to be generated by time-varying current distributions located at position $\boldsymbol{x}'$ and at the *past time* (or retarded time) $t_r = t - |\boldsymbol{x} - \boldsymbol{x}'|/c$.

As a final note, we point out that at Eq. (1.23), we specifically chose the retarded solution, in which the potentials at $(\boldsymbol{x}, t)$ depend on the sources at earlier times $t_r = t - |\boldsymbol{x} - \boldsymbol{x}'|/c$. This ensures that changes in the potentials caused by changes in the charge and current distributions propagate outward at the speed of light, encoding causality. The retarded potentials thus make explicit the *locality* encoded in the relativistic field theory.

One could, in principle, also construct the so-called *advanced potential*, which depend on sources at future times. The d'Alembert equation is time symmetric, so it's not surprising we get both kinds of solutions. The retarded potential is usually the useful one: given some distribution of charges or currents in the past, we can determine the potentials and fields they imply with the retarded solutions. Since the theory is reversible, if we instead know the "future" (or final states) of the charges, we can use the advanced potential to work out the prior potentials that would imply them.

We may also explicitly write down the retarded and advanced Green's functions, which solve

$$\partial^2 G(x - x') = \delta^{(4)}(x - x').$$

The two relevant solutions, the retarded (+) and advanced (−) Green's functions,

$$G_\pm(x - x') = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|}\delta\big(t - t' \mp |\boldsymbol{x} - \boldsymbol{x}'|/c\big). \tag{1.27}$$

I don't derive these explicitly in this case, though it can be seen easily that these may be integrated with the source term in Eq. (1.18) to produce the retarded and advanced potentials, respectively. Note that since $|\boldsymbol{x} - \boldsymbol{x}'| > 0$, the retarded Green's function is non-zero only for times $t > t_r$ (and converse for the advanced function).

1.3.2 Retarded (and advanced) Green's functions: Fourier method

Here, we shall again find the retarded (and advanced) Green's functions, this time using the method of Fourier transforms and complex integration. Here, we will work directly in the covariant form, and set $c = 1$, and use unbolded symbols for four-vectors, with $x = (t, \boldsymbol{x})$, and $x \cdot y = x_\mu y^\mu$.

Starting from the inhomogeneous d'Alembert equation (1.18), we associate the Green's function:

$$\partial^2 G = \delta^{(4)}(x - x'), \tag{1.28}$$

with Fourier representation:

$$G(x - x') = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\widetilde{G}(k)e^{-ik\cdot x}$$

(note the sign convention for the 4D Fourier transform). Plugging through, we find

$$\widetilde{G}(k) = \frac{-1}{k \cdot k}, \tag{1.29}$$

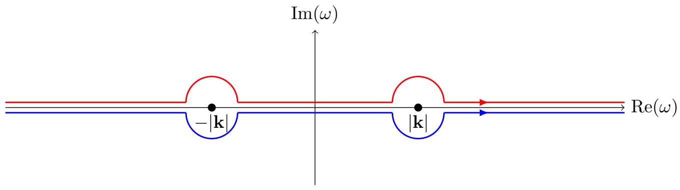

Figure 1.1: Contour plot for the ω integral in Eq. (1.31), where we have extended ω into the complex plane.

and taking the inverse Fourier transform:

$$G(x - x') = -\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{e^{-ik\cdot x}}{k \cdot k} \tag{1.30}$$

$$= \frac{-1}{2\pi}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\int \mathrm{d}\omega \frac{e^{-i\omega(t-t')}}{\omega^2 - \boldsymbol{k}^2}, \tag{1.31}$$

where $\omega \equiv k_0$. We first try to evaluate the ω integral, by extending into the complex plane and using the Cauchy residue theorem.

Note that there are two simple poles, at $\omega = \pm|\boldsymbol{k}|$, as shown in Fig. 1.1. These are along the real axis, right in the way of our contour. For the integral to converge, we will have to shift the contour to avoid the poles. It is not obvious that this is a remotely sensible thing to do; this integral doesn't converge, and contorting the contour changes the value of the integral! However, at the end, we can always check if what we have found is a valid solution to the Green's function equation (1.28), so for now we shall steam ahead.

There are several choices for avoiding the poles: for example, we can go above them (red contour in Fig. 1.1), or below them (blue contour). Let's first consider the red "above" contour. For reasons that will become clear in a moment, we shall consider two cases of $t > t'$ and $t < t'$ separately; the final solution is then the sum with appropriate theta functions:

$$G_{\text{total}} = G_{t>t'}\theta(t - t') + G_{t<t'}\theta(t' - t).$$

For $t > t'$, Jordan's Lemma tells us the integral (1.31) ($\sim e^{-i\omega|\delta t|}$) goes to zero at infinity in the lower plane, so we close the contour below. Both poles are enclosed, so we find:

$$\oint \mathrm{d}\omega \frac{e^{-i\omega(t-t')}}{\omega^2 - \boldsymbol{k}^2} = -2\pi i\left(\frac{e^{-i|\boldsymbol{k}|(t-t')} - e^{i|\boldsymbol{k}|(t-t')}}{2|\boldsymbol{k}|}\right)$$

$$= -\frac{2\pi}{|k|}\sin(|\boldsymbol{k}|(t - t')),$$

where the extra negative sign stems from this being the clockwise contour. For $t < t'$, on the other hand, Jordan's Lemma tells us to close the contour in the upper plane, in which neither pole is enclosed, and the integral is zero.

Therefore, we have:

$$G_r(x - x') = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\frac{\sin(|\boldsymbol{k}|(t - t'))}{|\boldsymbol{k}|}\theta(t - t'). \tag{1.32}$$

The 'r' subscript is to remind us that this was the solution from choosing the '$r$ed' contour; as we shall see in a moment, we can also realise this is will be the $r$etarded Green's function (note that is it nonzero only for "later" times $t > t'$). Writing $\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}') = |\boldsymbol{k}| r\cos\theta$, with $r = |\boldsymbol{x}-\boldsymbol{x}'|$, the integration over angles is performed similarly to Eq. (1.15), and we find:

$$G_r(x-x') = \frac{\theta(\Delta t)}{\pi\, r}\int_0^\infty \frac{\mathrm{d}|\boldsymbol{k}|}{2\pi}\sin(|\boldsymbol{k}|r)\sin(|\boldsymbol{k}|\Delta t). \quad (1.33)$$

Expanding the sin functions using Euler's formula, and making the change of variables $|\boldsymbol{k}| \to -|\boldsymbol{k}|$ for two of the terms (allowing us to extend the integral to $\pm\infty$), we find

$$G_r(x-x') = \frac{\theta(\Delta t)}{4\pi\, r}\int_{-\infty}^\infty \frac{\mathrm{d}|\boldsymbol{k}|}{2\pi}\left[e^{i|\boldsymbol{k}|(\Delta t - r)} - e^{i|\boldsymbol{k}|(\Delta t + r)}\right]. \quad (1.34)$$

The integrals are just the Fourier representation of the 1D delta functions, and only the first survives (since $\Delta t > 0$, enforced by the θ term, and $r > 0$ by definition). As such, we finally find:

$$G_r(x-x') = \frac{1}{4\pi\, r}\delta(t-t'-r), \quad (1.35)$$

which is the retarded Green's function (we dropped the θ term, since the delta function, which implies $t = t'+r$, already forces $\Delta t = t-t' > 0$).

In exactly the same way, if we choose the blue "lower" contour, we would find the *advanced* Green's function:

$$G_a(x-x') = \frac{1}{4\pi\, r}\delta(t-t'+r). \quad (1.36)$$

These can be integrated with the source term in the inhomogeneous d'Alembert equation (1.18) to immediately yield the advanced and retarded potentials.