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Symmetries, Conservation Laws, Noether's Theorem

Recap:

Classical non-relativistic $(q, \dot{q})$

General, infinitesimal transformation: $q \to q + \delta q$

$$\delta q_i = f_i(q)\, \varepsilon \quad \leftarrow \text{arbitrary infinitesimal}$$

$$\Rightarrow \delta \dot{q}_i = \frac{d}{dt}(\delta q_i) = \delta \dot{q}_i$$

$$\delta L = \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

$$\boxed{p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \qquad \text{E.L.:} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}, \quad \frac{d}{dt} p_i = \frac{\partial L}{\partial q_i}}$$

$$\Rightarrow \delta L = \sum_i \left(\dot{p}_i \delta q_i + p_i \delta \dot{q}_i \right)$$

$$= \frac{d}{dt}\left(\sum_i p_i \delta q_i \right)$$

$$= \frac{d}{dt}\underbrace{\left(\sum_i p_i f_i(q) \right)}_{Q} \varepsilon = \varepsilon \frac{d}{dt} Q$$

If this transform is a symmetry, $\Rightarrow \delta L = 0$

$$\Rightarrow \frac{d}{dt} Q = 0 \quad \Rightarrow \quad \sum_i p_i f_i(q) \text{ is conserved.}$$

E.g. Translation: $q_i \to q_i + \varepsilon$ ← const

$$Q = \sum_i p_i \quad \Rightarrow \text{ momentum } \left(p = \frac{\partial L}{\partial \dot q}\right) \text{ conservation!}$$

Rotation:

E.g.
$\delta x = -\delta\theta y, \quad \delta y = d\theta x$
$\delta\vec\theta$ points along $\hat z$

$$\delta\vec x = \delta\vec\theta \times \vec x$$

$$\left(dq_i = \sum_{jk} \varepsilon_{ijk}\, \delta\theta_j\, q_k \right)$$

$$Q = \sum_i \varepsilon_{ijk}\, q_j\, p_k = \sum_i (\vec x \times \vec p)_i$$

$\Rightarrow$ Angular momentum

Do slower:
n: each particle
i,j,k: coord directions

$$\delta q^n_i = \left(\varepsilon_{ijk}\, \hat n_j\, q^n_k\right) d\theta$$

$$Q = \sum_n \sum_i p^n_i f_i = \sum_n \sum_{ijk} p^n_i \varepsilon_{ijk} \hat n_j q^n_k$$

$$= \sum_{nijk} \hat n_j \left(q^n_k p^n_i \varepsilon_{kij} \right)$$

$\hat n_j$: j direction unit vector

$$= \sum_{nj} \hat n_j \left(\vec q^n \times \vec p^n\right)_j$$

Define $Q_j = \sum_n (q^n \times p^n)_j \to j$ comp of $\vec L$

Time translation:

$$\delta L = \frac{\partial L}{\partial t}\delta t$$

$$H \equiv \sum_i p_i \dot q_i - L$$

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i}\dot q_i + \frac{\partial L}{\partial \dot q_i}\ddot q_i \right) + \frac{\partial L}{\partial t} = \sum_i \left(\dot p_i \dot q_i + p_i \ddot q_i \right) + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\sum_i p_i \dot q_i \right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \quad \frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

$\Rightarrow$ Energy conservation

What about relativistic fields?

Consider : $\phi \to \phi + \delta\phi$

$$\delta\phi = \varepsilon f$$

ε: continuous, infinitesimal

Corresponding change to Lagrangian is $\delta\mathcal{L}$

To be a symmetry: $\delta\mathcal{L} = 0$ or total divergence

$$\delta\mathcal{L} = \varepsilon\, \partial_\mu K^\mu \qquad *$$

for some K^μ

can swap: Deriv's commute!

Now, $$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a)$$

* Write as though scalar, but valid for any

$$\boxed{\text{E-L:} \quad \frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)}$$

$$\Rightarrow \quad \delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi)$$

$$\delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] \qquad *$$

$\delta\phi = \varepsilon f$

$$\Rightarrow \quad \boxed{\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\overset{f}{\delta\phi}}{\varepsilon} - K^\mu\right] = 0}$$

Conserved current: J^μ

$$\boxed{\partial_\mu J^\mu = 0 \qquad , \qquad J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} f - k^\mu}$$

$$\left(f = \frac{\delta\phi}{\varepsilon} \right)$$

$$\delta\phi = \varepsilon f$$

- Locally conserved current, J^μ
  "Noether" current

- Noethers theorem:
  For every continuous symmetry of the action/system there is a locally conserved current, J^μ

$\partial_\mu J^\mu = 0$ is a continuity equation:

$$\frac{\partial}{\partial t} J^0 = - \nabla \cdot \vec{J}$$

Can define $\quad Q \equiv \int d^3x \, J^0$

↳ Conserved "charge"

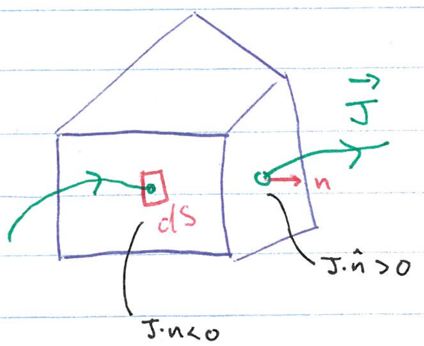
$$\frac{d}{\partial t} Q = - \int d^3x \, \nabla \cdot \vec{J}$$

Int. over finite Volume, V

$$\frac{d}{dt} Q_V = - \oint_S \vec{J} \cdot \hat{n} \, dS$$

S: surface of V

n: outward normal to S



Local conservation, much stronger than 'global' conserved Q from C.M.

Coordinate transformations

$$x^\mu \to x^\mu + \delta x^\mu$$

$$\delta\phi = (\partial_\mu \phi)\,\delta x^\mu \quad , \quad \delta\mathcal{L} = (\partial_\mu \mathcal{L})\,\delta x^\mu$$

Often, but not always: $\partial_\mu \delta x^\mu = 0$: No divergance

e.g., translations, rotations, not Boosts

Then: we can simplify:

$$\boxed{\delta\mathcal{L} = \partial_\mu (\underbrace{\mathcal{L}\,\delta x^\mu}_{\varepsilon k^\mu})}$$ for $\partial_\mu \delta x^\mu = 0$

Space-time Translations

Classically: (Non-rel) H, p conserved

$$x^\mu \to x^\mu + \delta x^\mu \quad , \quad \delta\phi = \underbrace{(\partial_\mu \phi)}_{f}\,\underbrace{\delta x^\mu}_{\varepsilon}$$

$$\delta x^\mu = \varepsilon^\mu \quad \to \text{constant}$$

$$\boxed{\begin{aligned} \delta\phi &= \partial_\mu \phi\, \delta x^\mu \\ f = \frac{\delta\phi}{\delta x^\mu} &= \partial_\mu \phi \end{aligned}}$$

$$\boxed{\begin{aligned} \varepsilon k^\mu &= \mathcal{L}\,\delta x^\mu \\ k^\nu &= \mathcal{L}\frac{\delta x^\nu}{\delta x^\mu} \\ &= \delta^\nu_\mu \mathcal{L} \end{aligned}}$$

For each μ, have conserved current

$$J^\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\nu \phi)}\partial_\mu \phi - \frac{\delta x^\nu}{\delta x^\mu}\mathcal{L}$$

$$\frac{\partial x^\nu}{\partial x^\mu} = \begin{cases} 1 & \nu = \mu \\ 0 \end{cases} \Rightarrow \delta^\nu_\mu$$

one for each μ:

$$\Rightarrow \boxed{T^\nu{}_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\nu \phi)}\partial_\mu \phi - \delta^\nu{}_\mu \mathcal{L}}$$

Stress-Energy Tensor

$\mu \leftrightarrow \nu$

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}}$$

Conserved if space/time translations are a symmetry

$$\partial_\mu T^{\mu\nu} = 0$$

Time translation:

$$\boxed{\nu = 0}$$

$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} + \partial_i T^{i0} = 0$$

$$\boxed{T^{00} = \frac{\partial \mathcal{L}}{\partial \dot\phi} \dot\phi - \mathcal{L} = \pi \dot\phi - \mathcal{L} = \mathcal{H} \; !}$$

$\mathcal{H}$: Energy density : "charge" conserved if time translation

$$T^{i0} = \frac{\partial \mathcal{L}}{\partial (\nabla \phi)_i} \dot\phi \qquad \text{- Energy density flux}$$

$$\frac{\partial \mathcal{H}}{\partial t} = -\nabla_i \cdot (T^{i0})$$

Space Translation $\nu = i \quad (1,2,3)$

Pressure / Stress / p-flux

$$\partial_\mu T^{\mu i} = \partial_0 \underbrace{T^{0i}}_{\text{Momentum density}} + \partial_j T^{ji} = 0$$

$$P^i \equiv \int d^3x \; T^{0i}$$

$$= \int d^3x \; \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \; \partial^i \phi$$

$$\vec{P} = -\int d^3x \; \pi \, (\nabla\phi)$$

"Charge" conserved if space-translation is symmetry!

- Conjugate momentum π : just conjugate variable
- P is physical momentum

Klein-Gordon

Very similar to our Quantised H

$$:H: = \int \frac{d^3k}{(2\pi)^3} \; \mathcal{E}_{\vec{k}} \; a^+_{\vec{k}} \, a_{\vec{k}}$$

$\left([a, a^+] = (2\pi)^3 \delta \quad \text{Normalisation} \right)$

Simple to show:

$$\boxed{:\vec{P}: = \int \frac{d^3k}{(2\pi)^3} \; \vec{k} \; a^+_{\vec{k}} \, a_{\vec{k}}}$$

Which supports interpretation of $\vec{k}$ as momentum of each particle!

Already Seen

$$\hat{P}\,|p\rangle = \hat{P}\, a^+_p |0\rangle = \int \frac{d^3k}{(2\pi)^3} \; \vec{k} \; a^+_{\vec{k}} \, a_{\vec{k}} \, a^+_p |0\rangle$$

$$a_k a_p^+ = \underbrace{[a_k, a_p^+]}_{(2\pi)^3\delta} + \underbrace{a_p^+ a_k}_{\Downarrow \; 0 \text{ on } |0\rangle}$$

$$= \int d^3k \; \vec{k} \; a^+_{\vec{k}} \, \delta(\vec{p}-\vec{k}) \, |0\rangle$$

$$= \vec{p}\,|p\rangle$$

$\Rightarrow$ $|p\rangle$ is a $\hat{p}$ eigenstate

Uniqueness of $T^{\mu\nu}$

If $\quad \partial_\mu T^{\mu\nu} = 0$

Notice that $\quad T^{\mu\nu} + \partial_\alpha K^{\mu\alpha\nu} \quad$ is also consered

if $\quad K^{\mu\alpha\nu} = - K^{\alpha\mu\nu}$, anti-symmetric in 2

$$
\begin{aligned}
\partial_\mu \partial_\alpha K^{\mu\alpha\nu} &= - \partial_\mu \partial_\alpha K^{\alpha\mu\nu} && \text{anti-symmetry} \\
&= - \partial_\alpha \partial_\mu K^{\alpha\mu\nu} && \text{deriv's commute} \\
&= - \partial_\mu \partial_\alpha K^{\mu\alpha\nu} && \mu \leftrightarrow \alpha \\
\Rightarrow \quad &= 0
\end{aligned}
$$

$T^{\mu\nu}$ not always symmetric, but is convienient to make it so!

Aside:

Example: Maxwell

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = (\vec{E}, \vec{B})$$

$$\mathcal{L} = \frac{-1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

$\rightarrow$ Can work out $T^{\mu\nu}$ directly: assignment

Can also 'guess'

What rank 2 tensors can we form from F that are 2^{nd} order in Field?

only 2:

$$T^{\mu\nu} \propto a\, F^{\mu\alpha} F_\alpha{}^\nu \quad + b\, \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

Find a and b from classical: $T^{00} = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)$

$$F^{\mu\alpha} F_\alpha{}^\nu \rightarrow F^{0\alpha} F_\alpha{}^0 = (-\vec{E}) \cdot (-\vec{E}) = \vec{E}^2$$

$$\Rightarrow \quad a = 1, \quad b = \frac{1}{4}$$

$$T^{\mu\nu} = F^{\mu\alpha} F_\alpha{}^\nu + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

Note: this not exactly the S.E tensor you get directly from Noether, but the symmetric version (Belinfante)

Lorentz Symmetries

Consider infinitesimal

Most places write ω

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon\, \lambda^\mu{}_\nu$$

For scalar, $\phi'(x) = \phi(\Lambda^{-1}x)$, $\Lambda^{-1}x \simeq 1 - \varepsilon\lambda$

$\delta\phi = -\varepsilon(\partial_\mu\phi)\lambda^\mu{}_\nu x^\nu$, $\delta\mathcal{L} = -\varepsilon(\partial_\mu\mathcal{L})\lambda^\mu{}_\nu x^\nu$

We discussed in week 1, Λ does not change η

$$\eta'^{\mu\nu} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\eta^{\alpha\beta} = \eta^{\mu\nu} \qquad (\eta' = \Lambda^T\eta\Lambda = \eta)$$

$$\eta^{\mu\nu} = \left(\delta^\mu{}_\alpha + \varepsilon\lambda^\mu{}_\alpha\right)\left(\delta^\nu{}_\beta + \varepsilon\lambda^\nu{}_\beta\right)\eta^{\alpha\beta}$$

$$= \eta^{\mu\nu} + \varepsilon\underbrace{\left(\lambda^{\nu\mu} + \lambda^{\mu\nu}\right)}_{=0} + O(\varepsilon^2)$$

$$\boxed{\begin{aligned}&\delta^\mu{}_\alpha\lambda^\nu{}_\beta\eta^{\alpha\beta}\\ &= \delta^\mu{}_\alpha\lambda^{\nu\alpha}\\ &= \lambda^{\nu\mu}\end{aligned}}$$

$\Rightarrow \lambda$ anti-symmetric

Note:
$$\partial_\mu\left(\lambda^\mu{}_\nu\, x^\nu\right) = \underbrace{\left(\partial_\mu\lambda^\mu{}_\nu\right)}_{=0}x^\nu + \lambda^\mu{}_\nu\,\overbrace{\partial_\mu x^\nu}^{=\delta^\nu_\mu}$$

0, since λ indep of position

$\lambda^\mu{}_\nu\delta^\nu_\mu = 0$, since λ anti-symm

$\Rightarrow$ zero diagonals!

$$\Rightarrow \quad \delta\mathcal{L} = -\varepsilon\left(\partial_\mu\mathcal{L}\right)\lambda^\mu{}_\nu x^\nu$$

$$= -\varepsilon\,\partial_\mu\Big(\underbrace{\lambda^\mu{}_\nu x^\nu \mathcal{L}}_{K^\mu}\Big)$$

$\Rightarrow$ Conserved Curr

$$J^\alpha = -\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}(\partial_\mu\phi)\lambda^\mu{}_\nu x^\nu + \lambda^\alpha{}_\nu x^\nu\mathcal{L}$$

Factor λ: (just numbers), and x

$$J^\alpha = -\lambda^\mu{}_\nu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} \partial_\mu \phi - \delta^\alpha_\mu \mathcal{L} \right)}_{T^\alpha{}_\mu} x^\nu$$

$$J^\alpha = -\lambda_{\mu\nu} T^{\alpha\mu} x^\nu$$

$$= +\frac{1}{2} \underbrace{\lambda_{\mu\nu}}_{\substack{6 \text{ parameters:} \\ 3 \text{ boosts} \\ 3 \text{ Rot's}}} \Big[\underbrace{x^\mu T^{\alpha\nu} - x^\nu T^{\alpha\mu}}_{M^{\alpha\mu\nu}} \Big]$$

From anti-symmetry

consider charge $(\alpha = 0)$ for rotations $(\mu, \nu = 1,2,3)$

T^{0i}: momentum density

$p^j = \int d^3x \, T^{0j}$

define $$M^{ij} = \int d^3x \, M^{0ij}$$

$$= \int d^3x \Big[\underbrace{x^i T^{0j} - x^j T^{0i}}_{\text{angular momentum density!}} \Big]$$

Charge for boosts, $\int d^3x \; x^i T^{00}$

$\sim$ center-of-mass^E, conservation

- Not scalar? Two Contributions to M : internal (spin) + external ('orbital')

Internal Symmetries

- Not all are coordinate

E.g., complex scalar field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - V(|\phi|^2)$$

Invariant under $\phi \to \phi e^{-i\alpha}$, $\phi^* \to \phi^* e^{i\alpha}$ (complex phase rotation)

$$\delta \mathcal{L} = 0$$

$$\delta\phi = -i\phi\,\delta\alpha, \qquad \text{(infinitesimal)}$$

$$\boxed{J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} f - k^\mu}$$

$\delta \mathcal{L} = \varepsilon \partial_\mu k^\mu$

$\delta\phi = \varepsilon f$

For each field, ϕ!

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(-i\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}(i\phi^*)$$

$$= -i(\partial^\mu \phi^*)\phi + i(\partial^\mu \phi)\phi^*$$

$[\phi, \phi^*] = 0$

$$J_\mu = i\left(\phi^* \partial_\mu \phi - (\partial_\mu \phi^*)\phi\right)$$

$$= i\,\phi^* \overleftrightarrow{\partial_\mu} \phi \qquad \text{Just notation}$$

$$Q = i\int d^3x\,\left(\phi^* \pi^* - \phi\,\pi\phi\right)$$

- Charges, Particle number, etc.

Conserved charges + Generators of transforms

Infinitesimal:

$$\phi \to \phi + \varepsilon f$$

$$= \phi_a + \varepsilon_b f_a(\phi_a, \partial_\mu \phi_a) \leftarrow \text{more general}$$

Symmetry label

Field label

Conserved Current $\quad J^{\mu b} = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} f^b_a - K^{\mu b}$

$$\left(\begin{aligned} \partial \mathcal{L} &= \varepsilon \, \partial_\mu K^\mu \\ &= \varepsilon_b \, \partial_\mu K^{\mu b} \end{aligned} \right)$$

Drop a, b now

Conserved Charge: $\quad Q^b = \int d^3x \, J^{0b} \quad$ (1 for each symmetry, b)

QFT: Q is quantum operator

(already saw $\hat{H}$, and $\hat{P}$)

Q's are generators of symmetries

$$\boxed{[Q, \phi] = -if}$$

or $\quad [\varepsilon_b Q^b, \phi_a] = -i \varepsilon_b f^b_a \quad$ (more general)

(same as classical case, with $-i[\,,\,] \to \{\,,\,\}_{PB}$)

$\partial \mathcal{L} = \varepsilon \partial_\mu K^\mu$

$\delta\phi = f\varepsilon$

Case 1:

Simplest case: f does not contain $\partial_t \phi$

and $K^0 = 0$

(e.g., spatial translations) $T^{\mu\nu} \to T^{\mu i}$

Also, all Internal symmetries (since don't involve coords!)

Then: $J^{0b} = \dfrac{\partial \mathcal{L}}{\partial(\dot{\phi}_a)} f_a^b = \pi_a f_a$

commutes, since

$$[Q, \phi] = \int d^3x \, [\pi(x) f(x), \phi(y)]$$

take $t=0$

$[\phi, \pi] = i\delta^{(3)}(\vec{x}-\vec{y})$

$$= \int d^3x \, (-i)\delta(x-y) f$$

$$\boxed{[Q, \phi(\vec{y})] = -i f(\vec{y})}$$

Case 2: Time translation

$\delta\phi = \underbrace{\dfrac{\partial\phi}{\partial t}}_{f!} \delta t$ does contain $\partial_t\phi$

Conserved Charge (density) : $T^{00} = \mathcal{H} = \pi \partial_t \phi - \mathcal{L}$

$$Q = H = \int d^3x \, \mathcal{H}$$

Now, simply have $[H, \phi] = -i \dfrac{\partial\phi}{\partial t}$ by Heisenberg

So, also true in this case!

Finite: Build-up as normal

$$\mathcal{U}_Q(\Omega_b) = e^{i\Omega_b Q^b}$$

Operator/Generator of transform!

Finite Parameters

On field:

$$\boxed{\phi_a' = \mathcal{U}_Q(\Omega_b)\,\phi_a\,\mathcal{U}_Q^{+}(\Omega_b)}$$

Note that for $\Omega \to \varepsilon$ (infinitesimal)

$$\mathcal{U} \simeq 1 + i\varepsilon_b Q^b$$

$$\delta\phi = i\varepsilon\,[Q,\phi] \quad , \text{ just as before.}$$

Aside

Clasically: Poisson bracket

F, G : any func. of can. variables p_i, q_i

$$\{F, G\} = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right)$$

$$\{q_i, p_j\} = \delta_{ij} \quad , \quad \{q_i, q_j\} = \{p_i, p_j\} = 0$$

(1) $$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t}$$

(2) $$\{F, q\} = -\frac{\partial F}{\partial p} \quad , \quad \{F, p\} = \frac{\partial F}{\partial q}$$

Back to Noether symmetry: $q \rightarrow q + \varepsilon f(q)$

Conserved: $Q = p_i f_i(q)$

change in F ?

$$\delta F = \frac{\partial F}{\partial q} \delta q$$

$$= \{F, p\} \varepsilon f$$

$$\delta F = \varepsilon \{F, Q\}$$ for $f(q) = f$ ↑ no q

To QM ? $\{A, B\} \longrightarrow [A, B] = i\hbar \{A, B\}$

$$\boxed{\varepsilon [F, Q] = i \delta F}$$ same!

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Recap: Symmetries + Noether currents

$$\phi_a \to \phi_a + \delta\phi_a \,, \qquad \delta\phi_a = \varepsilon_b f_a^b$$

$$\text{or} \quad \delta\phi_a = \underbrace{\frac{\partial \phi_a}{\partial \alpha_b}}_{f} \underbrace{\delta\alpha_b}_{\varepsilon}$$

a: field index

b: transform index

ε: infinitesimal parameter

If a symmetry, have conserved current J^μ: $\quad \partial_\mu J^\mu = 0$

$$\boxed{J^{\mu b} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \underbrace{\frac{\partial \phi_a}{\partial \alpha_b}}_{f} - K^{\mu b}}$$

$$\boxed{\begin{aligned} \delta\mathcal{L} &= \varepsilon_b \partial_\mu K^{\mu b} \\ &= \delta\alpha_b \, \partial_\mu K^{\mu b} \\ \text{or} \quad \partial_\mu K^{\mu b} &= \frac{\partial \mathcal{L}}{\partial \alpha_b} \end{aligned}}$$

Example: Translation $\quad x^\mu \to x^\mu + \varepsilon^\mu \,, \quad \varepsilon$ a const

$$\delta\phi = \underbrace{(\partial_\mu \phi)}_{f} \underbrace{\varepsilon^\mu}_{\varepsilon}$$

$$\Rightarrow \quad \boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}}$$

[ν: transform index

ν = 0 → time

ν = 1 → x

etc.]

Conserved charge(s)

$$Q^\nu \equiv \int d^3x \, T^{0\nu} = (H, \vec{P}) = P^\mu$$

→ are operators in QFT

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \quad \longrightarrow \quad H = \int \frac{d^3k}{(2\pi)^3} \varepsilon_{\vec{k}} \, a_k^\dagger a_k$$

$$P = -\int d^3x \, \pi (\nabla\phi) \quad \to \quad \vec{P} = \int \frac{d^3k}{(2\pi)^3} \vec{k} \, a_k^\dagger a_k$$

$$\Rightarrow \quad \hat{P}^\mu = \int \frac{d^3k}{(2\pi)^3} k^\mu a_k^\dagger a_k \quad \Big| \; k^0 = \varepsilon_{\vec{k}} \,, \text{ on shell}$$

Lorentz:

anti-symmetric

$$\Lambda \to 1 + \varepsilon \lambda$$

Conserved

$$J^{\alpha} = +\frac{1}{2} \underbrace{\lambda_{\mu\nu}}_{\text{Parameters of transform ("b")}} \left[x^{\mu} T^{\alpha\nu} - x^{\nu} T^{\alpha\mu} \right]$$

6 non-zero independant $\mu\nu$

$\Rightarrow$ 6 indep. conserved currents

$$J^{\alpha\mu\nu} \equiv x^{\mu} T^{\alpha\nu} - x^{\nu} T^{\alpha\mu} \quad , \quad \partial_{\alpha} T^{\alpha} = 0$$

Charges

$$M^{\mu\nu} = \int d^3x \left[J^{0\mu\nu} \right]$$

$$= \int d^3x \left(x^{\mu} T^{0\nu} - x^{\nu} \underbrace{T^{0\mu}}_{p^{\mu} \text{ density}} \right)$$

Example : Rotations : only spatial : $\mu, \nu = 1, 2, 3$

Rotation around z (3) axis

λ ?

$$\Lambda^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\theta \to \delta\theta, \quad \Lambda \to 1 + \delta\theta \lambda$$

$$\Lambda = \begin{pmatrix} 1 & & & 0 \\ & 1 & -\delta\theta & \\ & \delta\theta & 1 & \\ 0 & & & 1 \end{pmatrix} \Rightarrow \lambda = \begin{pmatrix} 0 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix}$$

$\lambda_{12} = -\lambda_{21} = 1$: rotation around z!

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Transformations

* Interested in how physical systems change under transformations, including:

For example:

Boosts } Lorentz* } Poincare*
Rotations
Translations
Internal Symmetries } whatever
etc.

* orthochronous
Also: T, P : later

We know how coordinates change

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \text{– Lorentz}$$

$$x'^\mu = x^\mu + a^\mu \qquad \text{– Translations}$$

What about classical/quantum fields/operators/states?

* Active vs. passive transformations

* Representations of Lorentz/Poincare group

* Classical vs. quantum

Active vs. passive transformations

Passive : Don't modify the system, but just the variables we use to describe it

$$\phi \to \phi' \quad : \quad \phi = f(\phi')$$

Action: $\quad S[\phi] = S[f(\phi')] = S'[\phi']$

$\Rightarrow$ This is a symmetry if $\boxed{S'[\phi'] = S[\phi'] \quad \text{PASSIVE}}$

Active: Consider change to system itself

$$\phi \to \phi' : \quad \phi' = f(\phi) \qquad \text{explicitly change the fields}$$

$$S[\phi] \to S[\phi'] = S'[\phi]$$

$\Rightarrow$ This is a symmetry if $\boxed{S'[\phi] = S[\phi] \quad \text{Active}}$

Groups: Super basic

Groups : set + operation $\otimes$

→ Closed : $g_i \otimes g_j = g_k \in G$

→ Identity : $I \cdot g_i = g_i$

→ Inverse : $g \otimes g^{-1} = I$

→ Assos. : $(g_i \cdot g_j) g_k = g_i \cdot (g_j \cdot g_k)$

Continuous groups: Lie groups

↳ described by N continuous parameters

Representation of a group : specific realisation of operations that preserve group structure.

Matrix reps: set of nxn matrices, one for each group element

Example: 2D rotations : characterised by single continuous parameter, θ

(2D → 1 param? because rotations preserve $|x|^2$ ⇒ eliminates the d.o.f)

$\vec{a} \cdot \vec{a} = a^T a \rightarrow a^T M^T M a$

$\Rightarrow M^T = M^{-1}$

⇒det=1, eliminates d.o.f

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix}$$

Terminology: (subset)

Special (S) : determinant = 1

Orthogonal (O): $M^T = M^{-1} \Rightarrow M^T M = 1$

Unitary (U): $U^\dagger U = 1$

M_θ is $SO(2)$

Isomorphic to $U(1)$: $M_\theta = e^{-i\theta}$ acts on $\mathbb{C}^1$

Generators

→ Lie groups parameterised by parameters, e.g. θ

choose: limit $\theta \to 0$ $\quad M \to 1$

Define generator X (typically matrix)

$$M = e^{-i \underbrace{X}_{\text{generator}} \underbrace{\theta}_{\text{parameter}}}$$

* We've seen this in action before!

For infinitesimal θ

$$M = 1 - i\theta X$$

E.g. $U(1)$: simplest, $X = 1$

$SO(2)$?

$$\frac{\partial M}{\partial \theta} = -iX e^{-iX\theta} \xrightarrow{\lim \theta \to 0} -iX$$

also:

$$\frac{\partial M}{\partial \theta} = \begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad X = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$ — Pauli σ_2 matrix

Generators form a vector space, Lie Algebra

↳ finite set of generators ⇒ infinite group elements

Lie bracket:

$$[X_i, X_j] = \underbrace{i}_{\text{convention}} \underbrace{f_{ijk}}_{\text{'structure' constants}} X_k$$

since group is closed

Example: $SU(2)$

take 'doublet' field

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

ϕ are complex scalars

Take, just as example:

$$\mathcal{L} = (\partial_\mu \psi^\dagger)(\partial^\mu \psi) - m^2 \psi^\dagger \psi$$

Note: only involves terms $\sim \psi^\dagger \psi = \phi_1^* \phi_1 + \phi_2^* \phi_2$

- let M be 2×2 matrix, rotations in 'doublet space'
- All terms invariant under $\psi \to M\psi$

if:

$$\psi^\dagger \psi = \psi^\dagger M^\dagger M \psi$$

(and $\partial M\psi = M \partial \psi$)

$$\Rightarrow \quad M^\dagger M = I$$

$[SU(2) \otimes U(1)]$

$e^{i\theta}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

'reducible' member of $U(1)$

Rotation in $\mathbb{C}$ phase, not "doublet" space.

Find $SU(2)$ elements: (generators)

$$e^{i\theta^j X_j^\dagger} e^{-i\theta^j X_j} = 1$$

$$\Rightarrow X_j^\dagger = +X_j$$

A basis (X) is

$$\underset{\sigma_x}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}, \quad \underset{\sigma_y}{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}, \quad \underset{\sigma_z}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$[\sigma_i, \sigma_j] = 2i\, \epsilon_{ijk}\, \sigma_k$$

Example: translations

$$x \to x^\mu + \varepsilon^\mu \quad , \quad \phi \to \phi + \delta\theta$$

$$\delta\phi = \partial_\mu \phi \, \varepsilon^\mu$$

$$M(\phi) = (1 - i\vec{X}\cdot\varepsilon^\mu)\,\phi$$

equating: $\vec{X} = \underbrace{i\,\partial_\mu}$ generator of translations!

$\Rightarrow \quad i\frac{\partial}{\partial t}$ generator of time translations : $\hat{\varepsilon}$

$-i\nabla$ generator of spatial translations : $\hat{p}$

In general: conserved charge, Q, under symmetry S
is the generator of S transformations

Conserved charges + Generators of transforms

Infinitesimal:

$$\phi \to \phi + \varepsilon f$$

$$= \phi_a + \varepsilon_b f^b_a(\phi_a, \partial_\mu \phi_a) \quad \leftarrow \text{ more general}$$

Symmetry label

Field label

Conserved Current $\quad J^{\mu b} = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} f^b_a - k^{\mu b}$

$$\left(\begin{aligned} \partial \mathcal{L} &= \varepsilon\, \partial_\mu k^\mu \\ &= \varepsilon_b\, \partial_\mu k^{\mu b} \end{aligned} \right)$$

Drop a, b now

Conserved Charge: $\quad Q^b = \int d^3x\, J^{0b}$ $\quad$ (1 for each symmetry, b)

QFT: Q is quantum operator

(already saw $\hat{H}$, and $\hat{P}$)

Q's are generators of symmetries

$$\boxed{[Q, \phi] = -i f}$$

or $\quad [\varepsilon_b Q^b, \phi_a] = -i \varepsilon_b f^b_a$ $\quad$ (more general)

(same as classical case, with $-i[\,,\,] \to \{\,,\,\}_{PB}$)

$\partial\mathcal{L} = \varepsilon \partial_\mu K^\mu$

$\delta\phi = f\varepsilon$

Case 1:

Simplest case: f does not contain $\partial_t \phi$

and $K^0 = 0$

(e.g., spatial translations) $T^{\mu\nu} \to T^{\mu i}$

Also, all Internal symmetries (since don't involve coords!)

Then: $$J^{0b} = \frac{\partial\mathcal{L}}{\partial(\dot{\phi}_a)} f_a^b = \pi_a f_a$$

$$[Q, \phi] = \int d^3x \, [\pi(x) \underbrace{f(x)}_{\text{commutes, since}}, \phi(y)]$$

take $t=0$

$$[\phi, \pi] = i\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$= \int d^3x \, (-i)\delta(x-y)\, f$$

$$\boxed{[Q, \phi(\vec{y})] = -i f(\vec{y})}$$

Case 2: Time translation

$$\delta\phi = \underbrace{\frac{\partial\phi}{\partial t}}_{f!} \delta t$$ does contain $\partial_t\phi$

Conserved charge (density): $T^{00} = \mathcal{H} = \pi\,\partial_t\phi - \mathcal{L}$

$$Q = H = \int d^3x \, \mathcal{H}$$

Now, simply have $[H, \phi] = -i \frac{\partial\phi}{\partial t}$ by Heisenberg

So, also true in this case!

Finite: Build-up as normal

$$\mathcal{U}_Q(\Omega_b) = e^{-i\Omega_b Q^b}$$

Operator/Generator of transform!

Finite Parameters

On field/operators:

$$\boxed{\phi_a' = \mathcal{U}_Q(\Omega_b)^\dagger \phi_a \, \mathcal{U}_Q^*(\Omega_b)}$$

* ?

Note that for $\Omega \to \varepsilon$ (infinitesimal)

$$\mathcal{U} \simeq 1 - i\varepsilon_b Q^b$$

$$\Rightarrow \boxed{\delta\phi = i\varepsilon\,[Q,\phi]}$$, just as before.

*

$$\left(\begin{aligned} O'(x) &= \langle \psi' | \hat{O} | \psi' \rangle \\ &= \langle \psi | \mathcal{U}^\dagger \hat{O}\, \mathcal{U} | \psi \rangle \\ &\Rightarrow \hat{O}' \to \mathcal{U}^\dagger \hat{O}\, \mathcal{U} \end{aligned} \right)$$

Aside

Clasically: Poisson bracket

F, G : any func of can. variables p_i, q_i

$$\{F,G\} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right)$$

$$\{q_i, p_j\} = \delta_{ij} \quad , \quad \{q_i, q_j\} = \{p_i, p_j\} = 0$$

(1) $$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t}$$

(2) $$\{F, q\} = -\frac{\partial F}{\partial p} \quad , \quad \{F, p\} = \frac{\partial F}{\partial q}$$

Back to Noether symmetry: $q \to q + \varepsilon f(q)$

Conserved: $Q = p_i f_i(q)$

change in F ?

$$\delta F = \frac{\partial F}{\partial q}\delta q$$

$$= \{F, p\}\varepsilon f$$

$$\delta F = \varepsilon \{F, Q\}$$ for $f(q) = f$ no q

To QM ? $\{A,B\} \to [A,B] = i\hbar \{A,B\}$

$$\boxed{\varepsilon [F, Q] = i\delta F}$$ same!

Example for KG fields:

H: generator of time translations

$$\phi(t,\vec{x}) = e^{iHt}\,\phi(0,\vec{x})\,e^{-iHt}$$

Heisenberg: $$\frac{\partial \hat{O}}{\partial t} = i\,[H,\hat{O}]$$

For a, a^{+}: $$\frac{\partial a^{+}_{\vec{k}}(t)}{\partial t} = i\,[H, a^{+}_{\vec{k}}(t)]$$

where $a^{+}_{\vec{k}}(t) \equiv e^{itH}\,a^{+}_{\vec{k}}\,e^{-itH}$

With $$H = \int \frac{d^3k}{(2\pi)^3}\,\mathcal{E}_{\vec{k}}\,a^{+}_{k} a$$

can show: $$[H, a_{\vec{k}}(t)] = -\mathcal{E}_{\vec{k}}\,a_{\vec{k}}(t)$$

$$[H, a^{+}_{\vec{k}}(t)] = \mathcal{E}_{\vec{k}}\,a^{+}_{\vec{k}}(t)$$

$$\boxed{\left(\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\mathcal{E}_{\vec{k}}}}\left(a_{\vec{k}}\,u_{\vec{k}}(x) + cc\right)\right) \quad a \to a(t) \Rightarrow \phi(\vec{x}) \to \phi(t,\vec{x})!}$$

$$\Rightarrow \quad \frac{\partial a(t)}{\partial t} = -i\mathcal{E}_{\vec{k}}\,a(t)$$

$$a_k(t) = e^{-i\mathcal{E}_k t}\,a_k$$

Not surprising!

Can do the same with $\phi(x)$, P - generator of space translations!

$$\phi(\vec{x}) = e^{-i\vec{x}\cdot\vec{P}}\,\phi(0)\,e^{+i\vec{x}\cdot\vec{P}}$$

$$\Rightarrow \quad \phi(t,\vec{x}) = \phi(x) = e^{+ix_\mu P^\mu}\,\phi(0)\,e^{-ix_\mu P^\mu}$$

operators

From before: $\Omega = x$, $Q = P$

Poincaré Invariance for fields

Example: Klein-Gordon

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

Translation:

$$x \to x' = x + a$$
$$\phi \to \phi'(x) = \phi(x-a)$$

Check
$$\partial'_\mu \equiv \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu = \partial_\mu$$

$$\partial'_\mu(x^\nu) = \frac{\partial(x'^\nu - a^\nu)}{\partial x'^\mu} = \frac{\partial x'^\nu}{\partial x'^\mu} = \delta^\nu_\mu$$

$\Rightarrow$ $\phi'(x)$ also a solution!
(obvious, but checked explicitly)

Lorentz

∂^2 is scalar, so invariant.. let's show anyway

$$x \to x' = \Lambda x$$
$$\phi \to \phi'(x) = \phi(\Lambda^{-1} x)$$

Seen before: Now: explicit!

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu$$

$$\frac{\partial(\Lambda^{-1}x')^\nu}{\partial x'^\mu} = \frac{\partial([\Lambda^{-1}]^\nu{}_\lambda x'^\lambda)}{\partial x'^\mu} = \underbrace{[\Lambda^{-1}]^\nu{}_\lambda}_{\text{constant!}} \frac{\partial(x'^\lambda)}{\partial x'^\mu}$$
$$= [\Lambda^{-1}]^\nu{}_\mu$$
$$= \Lambda_\mu{}^\nu$$

$$\Rightarrow \partial'_\mu \partial^{\mu'} = (\Lambda_\mu{}^\nu \partial_\nu)(\Lambda^\mu{}_\sigma \partial^\sigma) = \partial_\mu \partial^\mu$$

$\Rightarrow$ $\phi'(x)$ is a solution!

Poincare: Quantum states

Some transformation operator

$$|\psi\rangle \to \mathcal{U}|\psi\rangle$$

A passive transformation should not change observables:

$$\mathcal{U}(\Omega)|\psi\rangle = |\underbrace{\Omega\psi}\rangle$$ Notation for transformed state

$$|\langle\phi|\psi\rangle|^2 = |\langle\Omega\phi|\Omega\psi\rangle|^2$$

$\mathcal{U}$ is unitary operator, Ω elements of transformation group
$\mathcal{U}(\Omega)$ is a representation of the group

Require:

$\mathcal{U}(\Omega_1)\mathcal{U}(\Omega_2) = \mathcal{U}(\Omega_1\Omega_2)$	Combine Correctly
$\mathcal{U}(1) = 1$	Identity
$\mathcal{U}(\Omega^{-1}) = \mathcal{U}(\Omega)^{-1}$	Inverse
$\mathcal{U}(\Omega)^{+} = \mathcal{U}(\Omega)^{-1}$	Unitary

↳ same properties as Poincaré!

Stone's theorem:

$$\mathcal{U}(\Omega) = e^{i b_\mu P^\mu + \frac{i}{2}\lambda_{\mu\nu}J^{\mu\nu}}$$

parameters, Finite Angles/boosts

generators of translations

generators of Lorentz (3 boost, 3 rotations) Anti-Symmetric!

$$= \mathcal{U}(T) + \mathcal{U}(\Lambda)$$

P, J: Hermetian operators on Hilbert space

Remember:

Translation: Conserved charge $\int T^{0\mu}d^3x = P^\mu$

Lorentz: $J^{\mu\nu} = \int M^{0\mu\nu}d^3x = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu})\,d^3x$

$$\boxed{\mathcal{U}(\Lambda)^{+}\phi(x)\,\mathcal{U}(\Lambda) = \phi'(x) = \phi(\Lambda^{-1}x)}$$