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Propagators in QFT

Reminder: non-rel QM

cosider state, α at time t_0.

$$\boxed{|\alpha, t\rangle = \mathcal{U}\,|\alpha, t_0\rangle} \quad , \quad \mathcal{U} = e^{-iH(t-t_0)}$$

↳ Time evolution operator (Unitary operator)

$$\Psi_\alpha(\vec{x}, t) \equiv \langle \vec{x} |\, \mathcal{U} \,| \alpha, t_0\rangle$$

$$= \sum_a \underbrace{\langle \vec{x} | a\rangle}_{u_a(\vec{x})} \underbrace{\langle a | \alpha, t_0\rangle}_{C_a^\alpha\text{, constant}} e^{-i\varepsilon_a(t-t_0)}$$

Exand over basis $\{|a\rangle\}$
- Any set of energy eigenstates
- i.e. eigenstates of any operator $\hat{a}|a\rangle = a|a\rangle$ with $[H, \hat{a}] = 0$

$$\Psi_\alpha(\vec{x}, t) = \sum_a c_a\, u_a(\vec{x})\, e^{-i\varepsilon_a(t-t_0)}$$

Also: $$c_a = \langle a | \alpha, t_0\rangle = \int d^3x\, \underbrace{\langle a | \vec{x}\rangle}_{u_a^*(\vec{x})} \underbrace{\langle \vec{x} | \alpha, t_0\rangle}_{\Psi(\vec{x}, t_0)}$$

Combine:

$$\Psi_\alpha(\vec{x}, t) = \sum_a \int d^3x_0\, u_a(\vec{x})\, u_a^\dagger(\vec{x}_0)\, \Psi(\vec{x}_0, t_0)\, e^{-i\varepsilon_a(t-t_0)}$$

$$= \int d^3x_0\, \underbrace{G(\vec{x}, t; \vec{x}_0, t_0)}_{\text{The propagator}}\, \Psi(x_0, t_0)$$

- Somewhat like extension of $\mathcal{U}$
- know G : know entirey dynamics

$$G(x, t, x_0, t_0) = \sum_a \langle \vec{x} | a\rangle \langle a | \vec{x}_0\rangle e^{-i\varepsilon_a(t-t_0)}$$

$$= \sum_a \langle \vec{x} | e^{-iH(t-t_0)} | a\rangle \langle a | \vec{x}_0\rangle$$

$$= \langle \vec{x} |\, \mathcal{U}(t, t_0) \,| \vec{x}_0\rangle$$

$$= \langle \vec{x}, t | \vec{x}_0, t_0\rangle$$

Schro:

$\hat{X}|x\rangle = x|x\rangle$

Heis: $x \to x(t)$

$\hat{X}(t)\,|x,t\rangle = x\,|x,t\rangle$

just a label, state does not evolve

(1) Note: For fixed x_0, $t_0 < t$

G obeys Schrödinger Equation

$$G = \sum_a \underbrace{\langle \vec{x}|a\rangle e^{-i(\varepsilon_a(t-t_0)/\hbar)}}_{\psi_a(\vec{x},t)} \underbrace{\langle a|\vec{x}_0\rangle}_{\text{just const}}$$

$$\left(H - i\hbar\frac{\partial}{\partial t}\right) G = 0 \qquad \text{for} \quad t > t_0, \quad x_0 \text{ fixed}$$

(2) $t \to t_0$, $G \to \delta$

$$\boxed{G(\vec{x},t_0,\vec{x}_0,t_0) = \langle \vec{x}|\vec{x}_0\rangle = \delta^{(3)}(\vec{x}-\vec{x}_0)}$$

Define

$$G_R = \langle \vec{x},t|\vec{x}_0,t_0\rangle\,\Theta(t-t_0)$$

Non-zero for $t > t_0$

$$= \sum_a u_a(\vec{x})\,u_a^{+}(\vec{x}_0)\,e^{-i\varepsilon_a(t-t_0)/\hbar}\,\Theta(t-t_0)$$

Sub into S.E:

$$\boxed{\left(\underbrace{H_x}_{\text{with } x \text{ (not } x_0) \text{ coords}} - i\hbar\frac{\partial}{\partial t}\right) G_R = \underbrace{-i\hbar\,\delta(t-t_0)}_{-i\hbar\frac{\partial}{\partial t}\left(\Theta(t-t_0)\right)}\;\delta^{(3)}(\vec{x}-\vec{x}_0)}$$

$$\sum_a \langle x|a\rangle\langle a|x_0\rangle e^{-i\varepsilon_a(t-t_0)} \quad \delta!$$

$$= \sum_a \langle x|a\rangle\langle a|x_0\rangle$$

$$= \langle x|x_0\rangle$$

$$= \delta(x-x_0)$$

* Is a Green's function

Is there a QFT analogue?

NRQM $G(x, x_0) = \langle x, t | x_0, t_0 \rangle$

→ Amplitude for perfectly localised state $|x_0\rangle$ to 'propogate' to $|x\rangle$

→ G is wf of particle at x, t, which was at x_0 at t_0

- For general states $\psi(x,t) = \int d^3x_0\, G(x,t,x_0,t_0)\, \psi(x_0, t_0)$

QFT: No way to naturally prodce purely localised states

$\vec{x}$: not operator/eigenvalue, but a label/parameter

also $\langle \vec{x} | \vec{y} \rangle = \delta^{(3)}(\vec{x} - \vec{y})$: not Lorentz scalar

However, something somewhat similar/analagous:

$$\boxed{|x\rangle = \phi(x)|0\rangle}, \quad \langle x| = \langle 0|\phi^+$$

(we consider real field, $\phi^+ = \phi$)

Now, x^μ again

$\vec{x}, t \to x^\mu$

$$G \equiv \langle x | y \rangle = \langle 0 | \phi(x)\phi(y) | 0 \rangle \equiv D(x-y)$$

$e^{-i\varepsilon_p t + i\vec{p}\cdot\vec{x}}$ w/ $\varepsilon_p = +\sqrt{\vec{p}^2 + m^2}$

$$= \langle 0 | \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\varepsilon_k \varepsilon_p}} \left(a_{\vec{p}} u_{\vec{p}}(x) + a^+_{\vec{p}} u^*_{\vec{p}}(x)\right)\left(a_{\vec{k}} u_{\vec{k}}(y) + a^+_{\vec{k}} u^*_{\vec{k}}(y)\right) |0\rangle$$

(only cross-terms survive, $\langle 0 | a^+ a^+ |0\rangle = 0$, $a|0\rangle = 0$)

$$= \langle 0 | \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\varepsilon_k \varepsilon_p}} \cdot u_{\vec{p}}(x)\, u^*_{\vec{k}}(y)\, \underbrace{a_{\vec{p}} a^+_{\vec{k}}}_{(2\pi)^3 \delta(\vec{p}-\vec{k})} |0\rangle$$

$$\boxed{D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\varepsilon_{\vec{p}}} e^{-i\varepsilon_{\vec{p}}(x_0 - y_0)} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}} \quad \sim K_0(mr) \text{ Mod. Bessel of 2nd kind...}$$

- For $t_x = t_y$? not $\delta(\vec{x} - \vec{y})$
- Nearly though
- Non-rel limit, $\varepsilon_p \simeq m$

then, it is a δ

Last week, we saw this $\sim e^{-mr}$. "Local" for $r > m$ — Correlation length — Compton wavelength!

Aside

Nb: Can define "wave function" like thing

Consider some general field state $|\psi\rangle$

e.g. $|\psi\rangle = \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} \psi(\vec{p}) \underbrace{a_p^+|0\rangle}_{|\vec{p}\rangle}$ or whatever

$\psi(x) \equiv \langle x|\psi\rangle = \langle 0|\phi(x)|\psi\rangle$

- often called wave function
  - ↳ not same as NRQM
  - $|\psi(x)|^2$ is not (in general) a good prob. distribution.
  - x is not eigen value.

simplest case $|\psi\rangle = |p\rangle = \sqrt{2\mathcal{E}_p}\, a_p^+|0\rangle$

then $\langle x|p\rangle = e^{-i\mathcal{E}_p t} e^{i\vec{p}\cdot\vec{x}}$ → on-shell plane wave

Sorry: Again, notation differs between Sources. Here at least simple

// Notation $D(x-y) \equiv G_+(x-y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle$

$$D(x-y) \equiv G_+(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$
$$\Delta(x-y) \equiv \langle 0|[\phi(x),\phi(y)]|0\rangle = D(x-y) - D(y-x)$$

$$\Delta_R \equiv G_R(x-y) = \theta(t-t')\,\Delta(x-y) \qquad t=x_0,\ t'=y_0$$

$$\Delta_A \equiv G_A(x-x') = -\theta(t'-t)\,\Delta(x-x')$$
$$= +\theta(t-t')\,\Delta(x'-x)$$

$$\Delta_F \equiv G_F(x-x') = \theta(t-t')\,D(x-x') + \theta(t'-t)\,D(x'-x)$$

Define

Also $= D(x,x')$

$$G_+(x,x') \equiv \langle 0|\,\phi(x)\,\phi(x')\,|0\rangle$$

- Two-point (vac) correlation function
- $\sim$ Amplitude for $x' \to x$
- Wightman functions (condensed matter)

$$G_-(x,x') \equiv \langle 0|\,\phi(x')\,\phi(x)\,|0\rangle$$

$\hookrightarrow$ $G_+ \neq G_-$, since $[\phi(x),\phi(x')] \neq 0$ for non-equal times in general

(though, is zero for any space-ilike separated x, x')

and 3 propagators (just definitions for now):

$$G_R = \theta(t-t')\,\underbrace{\langle 0|\,[\phi(x),\phi(x')]\,|0\rangle}_{\Delta(x,x') = \Delta(x-x') = G_+ - G_-} = \Delta_R$$

- "Retarded" Green's function

$$G_A = -\theta(t'-t)\;\Delta(x-x') = \Delta_A$$

- Advanced Green's function
- Similar to NRQM

also

$$G_F(x,x') = \theta(t-t')\,G_+ + \theta(t'-t)\,G_- = \Delta_F$$
$$= \theta(t-t')\,D(x,x') + \theta(t'-t)\,D(x',x)$$

- Feynman propagator
- Time-ordered correlation function
- $\sim$ Amplitude $x' \to x$ for $t > t'$
  $x \to x'$ for $t < t'$
  (i.e., always forwards in time)

Also sometimes written as:

$$G_F = \langle 0|\,\underbrace{T}_{}\,\phi(x)\,\phi(x')\,|0\rangle$$

$\hookrightarrow$ Time-ordering operator: puts earlier times to right

Klein-Gordon Propagators (mostly general)

Exactly the same as we saw for NRQM:

Since
$$(\partial^2 + m^2)\phi = 0$$

$$(\partial^2 + m^2)\, G_\pm = 0$$

$$(\partial^2 + m^2)\, \Delta = 0$$

Correlation Functions

$G_+(x,y) = \langle 0|\phi(x)\phi(y)|0\rangle$

$G_-(x,y) = G_+(y,x)$

$\Delta = \langle 0|[\phi(x),\phi(y)]|0\rangle$

And for any $G = G_F, G_R, G_A$

Green's functions!

$$\boxed{(\partial^2_{(x)} + m^2)\, G(x,x') = -i\,\delta^{(4)}(x-x')}$$

*

$-i$ is convention. Matches w/ NRQM propagator + But may be different elsewhere

- Note: For space-like separated points $\Delta = \langle 0|[\phi(x),\phi(y)]|0\rangle = 0$

$$\Rightarrow \quad G_+ = G_- = G_F$$
$$G_R = G_A = \Delta = 0$$

- Also: not proven directly, but easy to check:

Due to Lorentz + translation symmetry of $|0\rangle$

$$G(x,x') = G(x-x') = G\big(\underbrace{(x-x')^2}_{ds^2}\big)$$

Important, since $\{x,x'\}$: 8 components

but ds^2 : 1

Note: 3 (+more) G functions, only 1 equation?

How?? ⤳ Return soon!

Example:

$$G_+(x-x') = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\mathcal{E}_{\vec{k}}} e^{-i\mathcal{E}_{\vec{k}}(t-t_0') + i\vec{k}(\vec{x}-\vec{x}')}$$

We calc'd this before!

As we saw last week:

$$= \int \frac{d^4k}{(2\pi)^4} 2\pi \underbrace{\delta(k^2-m^2)}_{\text{enforce on shell}} \underbrace{\theta(k_0)}_{\text{enforce } k_0=+\sqrt{\dots}} e^{-ik(x-x')}$$

// Fourier!

$$\Rightarrow \boxed{\tilde{G}_+(K) = 2\pi\,\delta(k^2-m^2)\,\theta(k_0)}$$

For the Green's functions

$$\boxed{(\partial^2 + m^2)\, G(x-x') = -i\,\delta^{(4)}(x-y)}$$

(Could calc directly from before, or evaluate Green's function)

Can directly see Fourier transform $\partial^2 \to -p^2 = -p_0^2 + |\vec{p}|^2$

$$(-k^2+m^2)\,\tilde{G}(k) = -i$$

$$G(k) = \frac{+i}{k^2-m^2} = \frac{i}{k_0^2 - |\vec{p}|^2 - m^2}$$

k_0 is not $\mathcal{E}_{\vec{k}}$!

Not on shell. K here is simply Fourier variable!

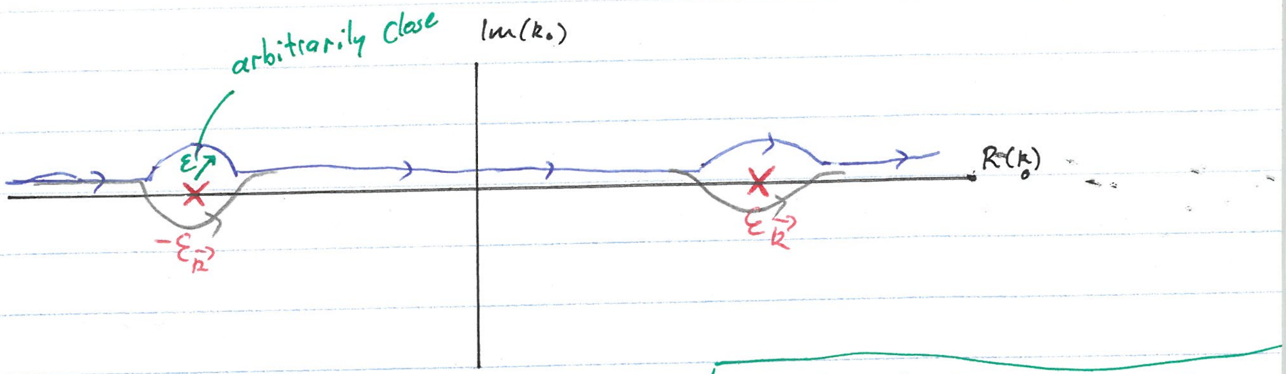
$$G(k) = \frac{i}{k_0^2 - \mathcal{E}_{\vec{p}}^2} = \frac{i}{\mathcal{E}^2 - \mathcal{E}_{\vec{p}}^2}$$

$$\boxed{G(x-x') = i\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-x')}}{k^2-m^2}}$$

• Poles: $\underset{k_0}{\mathcal{E}} = \pm \mathcal{E}_{\vec{k}}$, $\mathcal{E}_{\vec{k}} = +\sqrt{|\vec{k}|^2+m^2}$

Nb: bad poles: - along K contour, not inside
- Numerator $\neq 0$
(not like $\frac{\sin(x)}{x}$)

- This integral does not converge, is not well defined
- Can give meaning to integral by going around the poles "(Regularisation)" ↳ sort of
- Not immediately clear this is sensible...



→ 4 choices for going around poles

→ Will this give us the 3 $G_{R,A,F}$?

Let's do it:

> Note: In textbooks, it is common to define the Green's function first
> $(\partial^2+m^2)\,G = -i\delta$,
> then show the solutions are the propagators. Its nice to see it both ways

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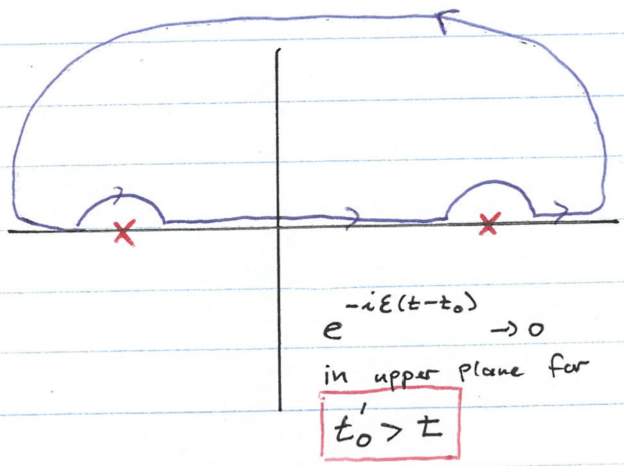
$\circledast$

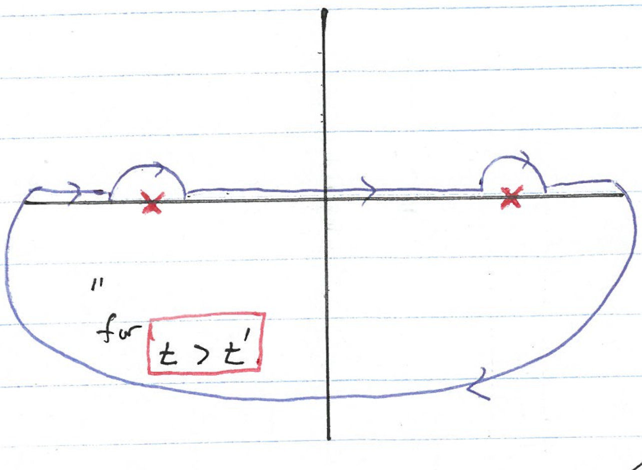
$\varepsilon \equiv k_0$ not $\varepsilon_{\vec{k}}$ (not on shell)

$$G(x-x') = i\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{d\varepsilon}{2\pi}\, \frac{e^{-i\varepsilon(t-t')}}{\varepsilon^2-\varepsilon_{\vec{k}}^2}$$

$$\left(\begin{aligned} &k^2-m^2 \\ &= k_0^2 - \underline{|\vec{p}|^2 - m^2} \\ &= k_0^2 - \varepsilon_{\vec{k}}^2 \end{aligned}\right)$$

1: Go above both : Should we close above or below?





For $t' > t$, close above. No poles + zero around circle:

$$\Rightarrow \quad G(x-x') = 0 \quad \text{for} \quad t' > t$$

For $t > t'$: close below. Two poles, zero around circle.

+ clockwise

$$\int \frac{d\varepsilon}{2\pi} \frac{e^{-i\varepsilon(t-t')}}{(\varepsilon+\varepsilon_{\vec{k}})(\varepsilon-\varepsilon_{\vec{k}})} = -\frac{2\pi i}{2\pi}\left[\frac{e^{-i\varepsilon_{k}(t-t')}}{2\varepsilon_{\vec{k}}} + \frac{e^{+i\varepsilon_{k}(t-t')}}{-2\varepsilon_{\vec{k}}}\right]$$

$$G(x-x') = \int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \times i \left(\quad \right)$$

$$= \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{x}')}}{2\varepsilon_{\vec{k}}}\left[e^{-i\varepsilon_{k}(t-t')} - e^{i\varepsilon_{k}(t-t')}\right]$$

* $\vec{k}$ is integration dummy
* $\varepsilon_{k} = \varepsilon_{-k}$
* therefore, in 2nd term, put $\vec{k} \to -\vec{k}$

$k\cdot x = k_\mu x^\mu$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\varepsilon_{\vec{k}}}\left(\underbrace{e^{-ik\cdot(x-x')}}_{D(x-x')} - \underbrace{e^{ik\cdot(x-x')}}_{D(x'-x)}\right)\Bigg|_{k_0 = \varepsilon_{\vec{k}}}$$

Back on shell!

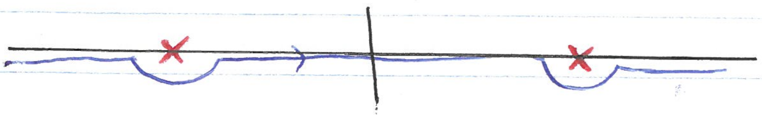
$$= \Delta(x-x')$$

Put together: We get the retarded Greens function!

$$\boxed{G_R = \Theta(t-t')\,\Delta(x-x')}$$

$$= G_R(x-x') = \Delta_R(x-x')$$

Other choice for contour : Advanced Green's Fn

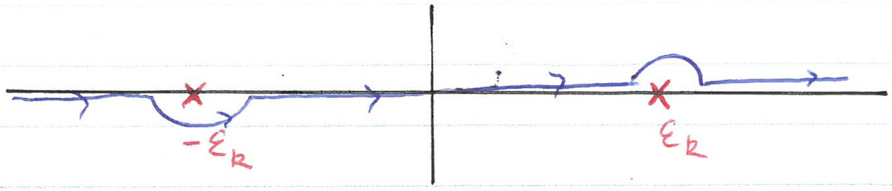


→ Exactly the same procedure, gives advanced Green's Fn :

$$\boxed{G_A(x-x') = \theta(t'-t)\,\Delta(x-x')} \qquad = \Delta_A(x-x')$$

- Similar interpretation to NRQM
- Retarded Green fn, G_R, non-zero only for ~~to~~ later times $t > t'$
- ~ Describes ~localised~ (nearest we can get in QFT) point solution to K-G equation that 'spawns' at x' and only affects points in the future
- Is zero for any space-like sep points (outside light cone)
- Easy enough to verify satisfies
$$(\partial^2 + m^2)\,G_{R,A} = -i\,\delta^{(4)}(x-x')$$
(we already saw this)

Yet another choice : Feynman Green's Function



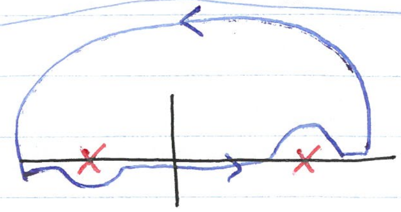
(Skip derivation, depending on time)

$\varepsilon = k_0$, not ε_k

Had: $$G(x-x') = i\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{d\varepsilon}{2\pi}\, \frac{e^{-i\varepsilon(t-t')}}{\varepsilon^2 - \varepsilon_{\vec{k}}^2}$$

For $t < t'$: upper circle $\to 0$

Pole $\varepsilon = -\varepsilon_K$



from $(i)^2$

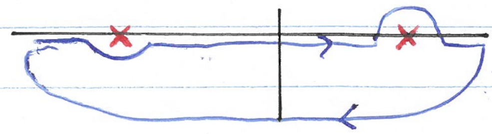
$$A \equiv i\int \frac{d\varepsilon}{2\pi}\, \frac{e^{+i\varepsilon_K(t-t')}}{(\varepsilon+\varepsilon_k)(\varepsilon-\varepsilon_k)} = -\left[\frac{e^{+i\varepsilon_K(t-t')}}{-2\varepsilon_K}\right]$$

again: put $\vec{k} \to -\vec{k}$ in $d^3\vec{k}$

$$G^{(t<t')} = \theta(t'-t)\, D(x'-x)$$

$$= \theta(t'-t) \int \frac{d^3k}{(2\pi)^3}\, \frac{1}{2\varepsilon_{\vec{k}}}\, e^{+ik\cdot(x-x')}\Big|_{k_0=\varepsilon_{\vec{k}}}$$

For $t > t'$: lower circle $\to 0$



pole @ $\varepsilon = \varepsilon_K$

– from $(i)^2$
– from clockwise

$$A = +\left[\frac{e^{-i\varepsilon_K(t-t')}}{2\varepsilon_{\vec{k}}}\right] \Rightarrow G^{(t>t')} = \theta(t-t') \int \frac{d^3\vec{k}}{(2\pi)^3}\, \frac{1}{2\varepsilon_{\vec{k}}}\, e^{-ik\cdot(x-x')}$$

$$= \theta(t-t')\, D(x-x')$$

Together:

$$\boxed{G_F(x-x') = \theta(t-t')\, D(x-x') + \theta(t'-t)\, D(x'-x)} = \Delta_F$$

$$= \langle 0|\, T\, \phi(x)\, \phi(x')\, |0\rangle$$

G_R: Propogates only to future

$G_+ = D(x-x')$: from $x' \longrightarrow x$
↳ Correlation between x, x'

G_F : For $t > t'$ from $x' \to x$
" $t < t'$ from $x \to x'$
} Time-ordered correlation function!

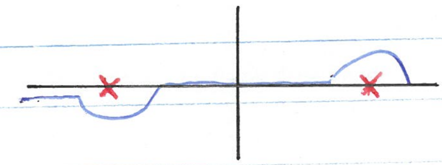
G_F: Plays central roll in interacting theories, perturbation theory + Feynman diagrams!

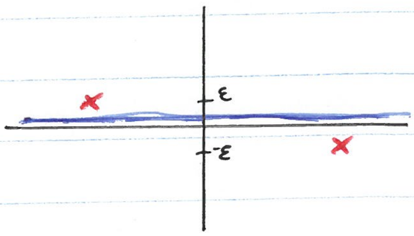
"$i\varepsilon$" convention

Notice we could equivalently shift the poles rather than the contour

poles: $k_0 = \mathcal{E} = \pm \mathcal{E}_p$

$\rightarrow \quad \mathcal{E} = \pm(\mathcal{E}_p - i\varepsilon)$ where $\varepsilon > 0$ infinitesimal





Simply a convenient trick. Simplifies matters:

$$\boxed{G_F(x-x') = i\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 - m^2 + i\varepsilon}}$$

No longer need to deform contour!

Often, require in momentum space only, which we can simply read

$$\boxed{\tilde{G}_F(k) = \frac{i}{k^2 - m^2 + i\varepsilon}}$$

Propagators - continued

Recap:

We defined, partly motivated by NRQM:

$$G_+(x-y) \equiv D(x-y) \equiv \Delta_+(x-y) = \langle 0| \phi(x)\phi(y) |0\rangle$$

→ Two pt. correlation function

→ Amplitude to propagate "point"-like losution $y \to x$

$$\boxed{G_+(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\mathcal{E}_{\vec{p}}} e^{-i\mathcal{E}_{\vec{p}}(x_0-y_0) + i\vec{k}\cdot(\vec{x}-\vec{y})}}$$

For large $|\vec{x}-\vec{y}|$, $G_+ \sim e^{-mr}$

Also defined

$$G_-(x-y) = \langle 0| \phi(y)\phi(x) |0\rangle$$

$$\Delta(x-y) = \langle 0| [\phi(x), \phi(y)] |0\rangle$$

And 3 Green's Functions:

$$G_R(x-y) = \theta(x_0-y_0)\,\Delta(x-y) = \Delta_R$$ * "Future" prop.

$$G_A(x-y) = -\theta(y_0-x_0)\,\Delta(x-y) = \Delta_A$$ * "Past" prop.

$$G_F(x-y) = \theta(x_0-y_0)\,G_+(x-y) + \theta(y_0-x_0)\,G_-(x-y)$$

$$= \langle 0| T(\phi(x)\phi(y)) |0\rangle$$

* Feynman prop
* Time-ordered
* Forwards-in-time propagation

Kilen-Gordon

All ✓ Green's functions satisfy

$$\boxed{(\partial^2 + m^2)\, G_{F,R,A}(x-y) = -i\, \delta^{(4)}(x-y)}$$

(proof at end of Notes)

From Fourier:

$$(-k^2 + m^2)\, \tilde{G}(k) = -i$$

(NB: Common to just write as $G(k)$, without $\sim$)

$$\Rightarrow G(k) = \frac{i}{k^2 - m^2}$$

Not on shell

Inverse Forier:

$$\Rightarrow \quad G(x-y) = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2}$$

$$= i \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_x - t_y)}}{\omega^2 - \varepsilon_{\vec{k}}^2}$$

$$\left(\begin{aligned} k^2 - m^2 &= k_0^2 - |\vec{k}|^2 - m^2 \\ &= k_0^2 - \varepsilon_{\vec{k}}^2 \\ &= \omega^2 - \varepsilon_{\vec{k}}^2 \end{aligned}\right)$$

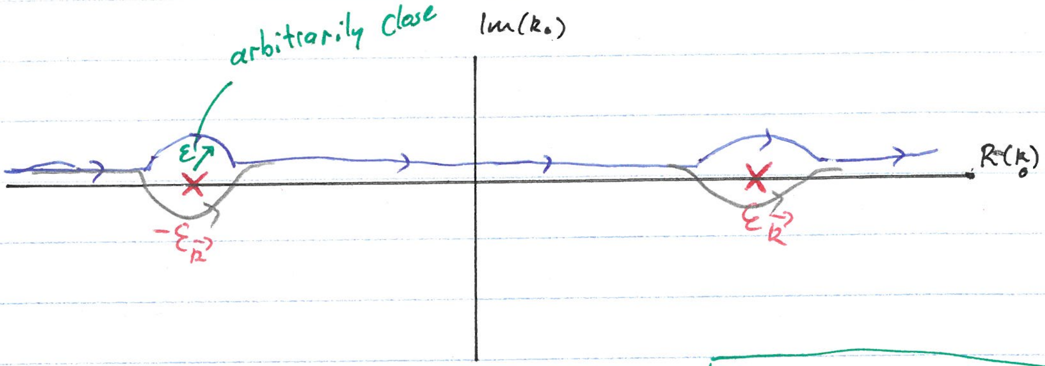
(Just notation!)

Poles at $\omega = \pm\, \varepsilon_{\vec{k}}$

* Bad poles, along contour, numerator $\neq 0$

(Sorry: I switch $\omega \to \varepsilon$ later)

- This integral does not converge, is not well defined
- Can give meaning to integral by going around the poles ("Regularisation") ↳ sort of
- Not Immediately clear this is sensible...



→ 4 choices for going around poles

→ Will this give us the 3 $G_{R,A,F}$?

Let's do it:

Note: In textbooks, it is common to define the Green's function first $(\partial^2+m^2)G = -i\delta$, then show the solutions are the propagators. Its nice to see it both ways

---

(*)

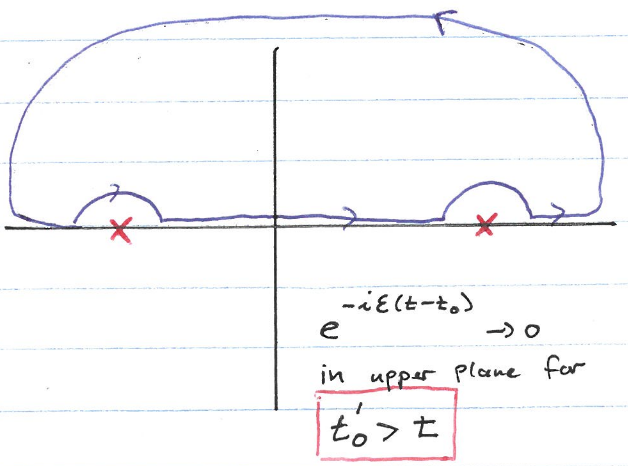
$$G(x-x') = i\int\frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\int\frac{d\varepsilon}{2\pi}\,\frac{e^{-i\varepsilon(t-t')}}{\varepsilon^2-\varepsilon_{\vec{k}}^2}$$

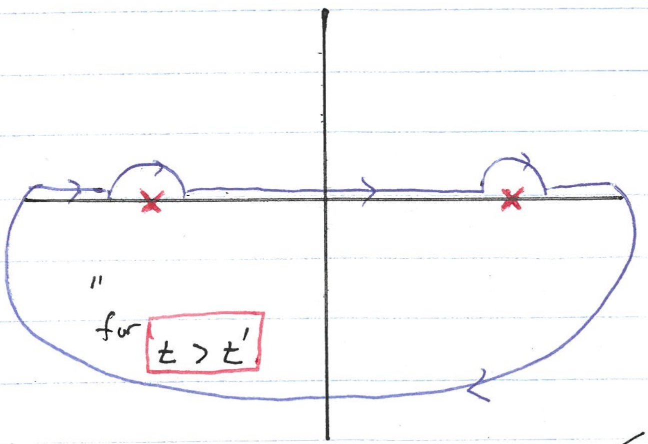
$\varepsilon \equiv k_0$ not $\varepsilon_{\vec{k}}$ (not on shell)

$$\left(\begin{aligned} &k^2-m^2 \\ &= k_0^2 - \underbrace{|\vec{p}|^2 - m^2} \\ &= k_0^2 - \varepsilon_{\vec{k}}^2 \end{aligned}\right)$$

$\varepsilon \equiv k_0$

1: Go above both: Should we close above or below?





For $t'>t$, close above. No poles + zero around circle:

$$\Rightarrow G(x-x') = 0 \quad \text{for} \quad t'>t$$

For $t>t'$: close below. Two poles, zero around circle.

+ clockwise

$$\int \frac{d\mathcal{E}}{2\pi} \frac{e^{-i\mathcal{E}(t-t')}}{(\mathcal{E}+\mathcal{E}_{\vec{k}})(\mathcal{E}-\mathcal{E}_{\vec{k}})} = -\frac{2\pi i}{2\pi}\left[\frac{e^{-i\mathcal{E}_k(t-t')}}{2\mathcal{E}_{\vec{k}}} + \frac{e^{+i\mathcal{E}_k(t-t')}}{-2\mathcal{E}_{\vec{k}}}\right]$$

$$G(x-x') = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \times i \left(\quad \right)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{x}')}}{2\mathcal{E}_{\vec{k}}}\left[e^{-i\mathcal{E}_k(t-t')} - e^{i\mathcal{E}_k(t-t')}\right]$$

* $\vec{k}$ is integration dummy
* $\mathcal{E}_k = \mathcal{E}_{-k}$
* therefor, in 2^{nd} term, put $\vec{k} \to -\vec{k}$

$k\cdot x = k_\mu x^\mu$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\mathcal{E}_{\vec{k}}}\left(\underbrace{e^{-ik\cdot(x-x')}}_{D(x-x')} - \underbrace{e^{ik\cdot(x-x')}}_{D(x'-x)}\right)\Bigg|_{k_0=\mathcal{E}_{\vec{k}}}$$

Back on shell!

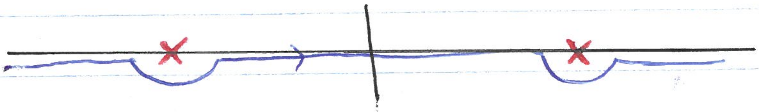
$$= \Delta(x-x')$$

Put together: We get the retarded Greens function!

$$\boxed{G_R = \Theta(t-t')\,\Delta(x-x')}$$

$$= G_R(x-x') = \Delta_R(x-x')$$

Other choice for contour : Advanced Green's Fn



→ Exactly the same procedure, gives advanced Green's Fn :

$$\boxed{G_A(x-x') = -\theta(t'-t)\,\Delta(x-x')} \qquad = \Delta_A(x-x')$$

- Similar interpretation to NRQM

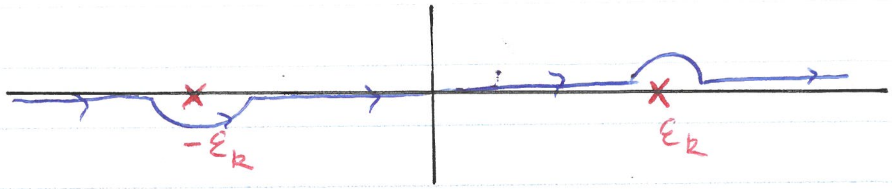
- Retarded Green fn, G_R, non-zero only for later times $t>t'$

- ~ Describes ~localised~ (nearest we can get in QFT) point solution to K-G equation that 'spawns' at x' and only affects points in the future

- Is zero for any space-like sep points (outside light cone)

- Easy enough to verify satisfies
$$(\partial^2+m^2)\,G_{R,A} = -i\,\delta^{(4)}(x-x')$$
(we already saw this)

Yet another choice : Feynman Green's Function

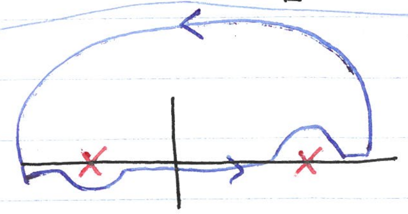


(Skip derivation, depending on time)

$\varepsilon = k_0$, not ε_k

Had: $$G(x-x') = i\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{d\varepsilon}{2\pi}\, \frac{e^{-i\varepsilon(t-t')}}{\varepsilon^2-\varepsilon_{\vec{k}}^2}$$

For $t<t'$: upper circle $\to 0$



pole $\varepsilon = -\varepsilon_k$

from $(i)^2$

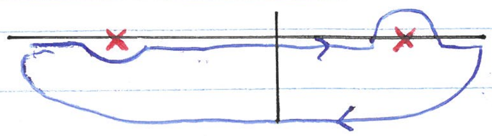
$$A \equiv i\int \frac{d\varepsilon}{2\pi}\, \frac{e^{+i\varepsilon_k(t-t')}}{(\varepsilon+\varepsilon_k)(\varepsilon-\varepsilon_k)} = -\left[\frac{e^{+i\varepsilon_k(t-t')}}{-2\varepsilon_k}\right]$$

(again: put $\vec{k}\to-\vec{k}$ in $d^3\vec{k}$)

$$G^{(t<t')} = \theta(t'-t)\; D(x'-x)$$

$$= \theta(t'-t)\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\varepsilon_{\vec{k}}}\, e^{+ik\cdot(x-x')}\Big|_{k_0=\varepsilon_{\vec{k}}}$$

For $t>t'$: lower circle $\to 0$



pole @ $\varepsilon = \varepsilon_k$

$-$ from $(i)^2$
$-$ from clockwise

$$A = +\left[\frac{e^{-i\varepsilon_k(t-t')}}{2\,\varepsilon_{\vec{k}}}\right] \Rightarrow G^{(t>t')} = \theta(t-t')\int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{2\varepsilon_{\vec{k}}}\, e^{-ik\cdot(x-x')}$$

$$= \theta(t-t')\,D(x-x')$$

Together:

$$\boxed{G_F(x-x') = \theta(t-t')\,D(x-x') + \theta(t'-t)\,D(x'-x)} = \Delta_F$$

$$= \langle 0|\,T\,\phi(x)\,\phi(x')\,|0\rangle$$

G_R: Propogates only to future

$G_+ = D(x-x')$: from $x' \to x$

↳ Correlation between x, x'

G_F : For $t > t'$ from $x' \to x$
" $t < t'$ from $x \to x'$ } Time-ordered correlation function!

G_F: Plays central roll in interacting theories, perturbation theory + Feynman diagrams!

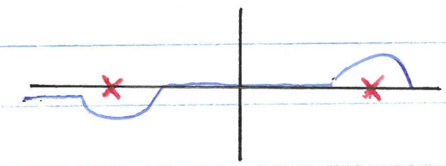
ε is a positive infinitesimal
Sorry: ε_k was a bad choice for k_0, $\varepsilon_{\vec{k}}$ ...

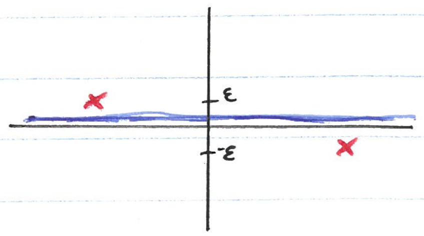
"$i\varepsilon$" convention

Notice we could equivalently shift the poles rather than the contour

poles: $k_0 = E = \pm E_{\vec{p}}$

$\rightarrow \quad E = \pm(E_p - i\varepsilon)$ where $\varepsilon > 0$ infinitesimal





Simply a convenient trick. Simplifies matters:

$$\boxed{G_F(x-x') = i\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 - m^2 + i\varepsilon}}$$

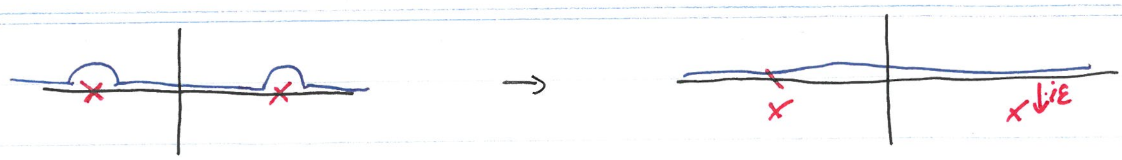
No longer need to deform contour!

Often, require in momentum space only, which we can simply read

$$\boxed{\tilde{G}_F(k) = \frac{i}{k^2 - m^2 + i\varepsilon}}$$

Retarded

G_R



poles @ $E = \pm E_k - i\varepsilon$

$$G_R(x-x') = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{(E+i\varepsilon)^2 - E_{\vec{k}}^2}$$

$E = k_0$, not $\mathcal{E}_{\vec{k}}$

$E_{\vec{k}} = |\vec{k}|^2 + m^2$

G_A: same, $+i\varepsilon \to -i\varepsilon$

Summary:

$$G_F(x-x') = \langle 0| T(\phi(x)\phi(x'))|0\rangle = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-x')}}{k^2-m^2+i\varepsilon}$$

$$\tilde{G}_F(k) = \frac{i}{k^2-m^2+i\varepsilon}$$

$$G_R^{(x-x')} = \Theta(t-t')\,\Delta(x-x') = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{(E+i\varepsilon)^2 - E_{\vec{k}}^2}$$

$$G_{R,A}(k) = \frac{i}{(E \pm i\varepsilon)^2 - E_{\vec{k}}^2}$$

$$\Delta(x-y) = \langle 0| [\phi(x), \phi(y)] |0\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\mathcal{E}_{\vec{k}}} \left[e^{-ik\cdot(x-y)} - e^{ik(x-y)} \right] \Bigg|_{k_0 = \mathcal{E}_{\vec{k}}}$$

$$= D(x-y) - D(y-x)$$

$$= G_+(x-y) - G_+(y-x)$$

<u>Quick look at Interacting theory</u>

↳ Semi-classical

Example from last week:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right) + \underbrace{\lambda\,\chi^2_{(x)}}_{J(x)}\,\phi(x)$$

Last time: ϕ, χ classical

Now: ϕ as Quantum scalar field $J(x) = \lambda\chi^2_{(x)}$: classical source

E.O.M: $$(\partial^2 + m^2)\,\phi(x) = J(x)$$

with (Retarded) Green's fn

$$(\partial^2 + m^2)\,G_R(x, x') = \underbrace{-i}_{\text{QM convention}}\,\delta^{(4)}(x-x')$$

General solution

$$\phi(x) = \phi_0(x) + \phi_*(x)$$

w/ $(\partial^2 + m^2)\,\phi_0(x) = 0$

↳ Non-interacting Sol^n

$$\phi_*(x) = i\int d^4y\; G_R(x-y)\,J(y)$$

Note: by use of G_R: if $J(x)$ was 0 in the past
then $\phi_* = 0$ so $\phi = \phi_0$

↝ ϕ_0 was "existing" solution

↳ ϕ_* was "produced" by $J(x)$

$$G_R(x-y) = \theta(\overset{x_0}{t_x} - \overset{y_0}{t_y})\,\Delta(x-y)$$

$$\Delta(x-y) = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\mathcal{E}_{\vec{k}}} \left(e^{-ik(x-y)} - e^{ik(x-y)} \right) \Big|_{k_0 = \mathcal{E}_{\vec{k}}}$$

on shell!

$$\phi_*(x) = i \int d^4y \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\mathcal{E}_{\vec{k}}} \left(e^{-ik(x-y)} - e^{ik(x-y)} \right) J(y)\, \theta(x_0 - y_0)$$

Let $J(y) = 0$ for $y_0 > x_0$

i.e. $J(y)$ only acts in the past

$\Rightarrow$ θ not required

$$\Rightarrow \quad \phi_*(x) = i \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\mathcal{E}_{\vec{k}}} \left(e^{-ik\cdot x} \tilde{J}(k) - e^{ikx} \tilde{J}^*(k) \right) \Big|_{k_0 = \mathcal{E}_{\vec{k}}}$$

where $$\tilde{J}(k) = \int d^4x\, e^{+ikx} J(x)$$

↳ can now write general solution for $\phi = \phi_0 + \phi_*$

$$\phi_0(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\mathcal{E}_{\vec{k}}}} \left(a_{\vec{k}} u_{\vec{k}}(x) + a^+_{\vec{k}} u^*_{\vec{k}}(x) \right)$$

$$\left(\text{w/ } [a_p, a_q^+] = (2\pi)^3 \delta(\vec{p} - \vec{q}) \right)$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} \left[\left(a_{\vec{k}} + \frac{i\tilde{J}(k)}{\sqrt{2\varepsilon_{\vec{k}}}} \right) u_{\vec{k}}(x) + cc \right]$$

$\Rightarrow$ effect of shifting the free field creation/annihilation operators!

$$a_k \to a_k + i\rho_k(k) \qquad \left(\rho_k = \frac{\tilde{J}(k)}{\sqrt{2\varepsilon_k}} \right) \Big|_{k_0 = \varepsilon_{\vec{k}}}$$

$$\mathcal{H} = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2 \right) - J\phi$$

Shift in Energy:

$$:H_0: = \int \frac{d^3k}{(2\pi)^3} \varepsilon_{\vec{k}} \, a^\dagger a$$

$$:H: = \int \frac{d^3k}{(2\pi)^3} \varepsilon_{\vec{k}} \left(a^\dagger_{\vec{k}} - i\rho_{\vec{k}}(k) \right)\left(a_{\vec{k}} + i\rho_{\vec{k}}(k) \right) - \int d^3\vec{x} \, J(x)\phi(x)$$

$$\langle 0 | H_0 | 0 \rangle = 0$$

$$\langle 0 | H | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{|\tilde{J}(k)|^2}{2} \quad \text{w/ } k_0 = \varepsilon_{\vec{k}} \qquad = \int \frac{d^3k}{(2\pi)^3} \varepsilon_{\vec{k}} \, \rho(k)$$

Created particles

$$N_0 = \int \frac{d^3k}{(2\pi)^3} a^\dagger_k a \qquad (\hat{n} = a^\dagger a \text{ from HO})$$

$$\langle 0 | N | 0 \rangle = \int \underbrace{\frac{d^3k}{(2\pi)^3} \frac{1}{2\varepsilon_{\vec{k}}}}_{\text{L. Invariant } \vec{p}\text{-vol element}} |\tilde{J}(k)|^2 \quad @ \ k_0 = \varepsilon_{\vec{k}} \qquad = \int \frac{d^3k}{(2\pi)^3} \rho(k)$$

* Source generates particles of density $|\tilde{J}(k)|^2$ per L.I p-volume
* Compare $\langle N \rangle$ to $\langle H \rangle$: each particle has energy $\varepsilon_{\vec{p}}$!

Extra

Proof that G_R is Green's fn:

$$(\partial^2+m^2)\, G_R = -i\,\delta^{(4)}(x-y)$$

$$\begin{aligned} G_R(x-x') &= \theta(t-t')\,\Delta(x-x') \\ &= \theta(t-t')\,\langle 0|\,[\phi(x),\phi(x')]\,|0\rangle \\ &= \theta(t-t')\,\big(G_+(x,x') - G_-(x,x')\big) \end{aligned}$$

$$(\partial^2+m^2)\, G_R = (\partial_t^2 - \nabla_x^2 + m^2)\, G_R$$

∇ acts on $\vec{x}$, not $\vec{x}'$, same for ∂_t

Short-hand $\theta = \theta(t-t')$, $\Delta = \Delta(x-x')$

$$= \partial_t^2(\theta\,\Delta) + \theta\,(-\nabla^2+m^2)\,\Delta(x-x')$$

$$= \ddot{\theta}\,\Delta + 2\dot{\theta}\dot{\Delta} + \theta\ddot{\Delta} + \theta\,(-\nabla^2+m^2)\,\Delta$$

$$= \ddot{\theta}\,\Delta + 2\,\dot{\theta}\dot{\Delta} + \theta\,\underbrace{(\partial^2+m^2)\,\Delta}_{=0}$$

In distributional sense, use I.B.P

$$\ddot{\theta}\,\Delta = -\,\dot{\theta}\dot{\Delta}$$

$$\Rightarrow \quad (\partial^2+m^2)\, G_R = \underbrace{\left(\frac{\partial}{\partial t}\theta(t-t')\right)}_{\delta(t-t')} \cdot \underbrace{\frac{\partial}{\partial t}\Delta(x-x')}_{-i\,\delta^{(3)}(\vec{x}-\vec{x}')}$$

$$\frac{\partial}{\partial t}\Delta = \frac{\partial}{\partial t}\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\,\mathcal{E}_{\vec{k}}}\left[e^{-i\mathcal{E}_k(t-t') + i\,\vec{k}\cdot(\vec{x}-\vec{x}')} - e^{ik(x-x')} \right]$$

$k_0 = \mathcal{E}_k$

$$= \int \frac{d^3k}{(2\pi)^3}\,\frac{-i\mathcal{E}_k}{2\,\mathcal{E}_{\vec{k}}}\left[e^{-ik(x-x')} + e^{ik(x-x')} \right]$$

We have $\delta(t-t')$, so $t=t'$, and $\vec{k} \to -\vec{k}$ in 2^{nd} term

$$= -i\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = -i\,\delta^{(3)}(\vec{x}-\vec{x}')$$

Extra #2

derivative of $\theta(t)$

$$F_\theta[g] \equiv \int_{-\infty}^{\infty} \theta(t)\, g(t)$$

$$F_{\theta'}(g) \equiv \int_{-\infty}^{\infty} \frac{\partial \theta(t)}{\partial t}\, g(t)$$ use I·B·P:

$$= -\int_{-\infty}^{\infty} \theta(t)\, \frac{\partial g(t)}{\partial t} + \cancel{\text{Boundary}}^{\,*\, 0}$$

$$= -\int_{0}^{\infty} \frac{dg(t)}{dt}\, dt$$ by definition of θ

$$= -\left[\cancel{g(\infty)}^{\,0\,*} - g(0) \right]$$

$$= g(0)$$

$$\Rightarrow \quad F_{\theta'}[g] = F_\delta[g]$$

* Since distribution defined for convergent test functions that $g(x) \to 0$ $x \to \pm\infty$. Our fields, ϕ, assumed to have the same property!