B.M. Roberts

PHYS4040: QFT - Lecture 5 - W3(A) - 2024

Notes: 2

* Distributions, delta 'functions', Green's functions, Complex integration

Motivation: consider two scalar fields (just classical for now)

$$\boxed{\mathcal{L} = \mathcal{L}_{free}(\phi) + \mathcal{L}_{free}(\chi) \underbrace{- \lambda\phi^2\chi}_{\text{interaction term}}}$$

$$\mathcal{L}_{free} = \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - m_\phi^2\phi\right]$$

$\mathcal{L}_{int} = -\lambda\phi^2\chi$

$\lambda = 0$: two independant K.G fields: no change from before

$\lambda \neq 0$: Interactions

previous E.O.M: $(\partial^2 + m_\phi^2)\phi = 0$

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)$$

Now: extra terms:

$$\boxed{\begin{aligned}(\partial^2 + m_\phi^2)\phi &= -2\lambda\phi\chi \\ (\partial^2 + m_\chi^2)\chi &= -\lambda\phi^2\end{aligned}}$$

* ϕ and χ are now coupled
* Before: $\phi = 0 \rightarrow$ stays zero
* Now: $\phi = 0$, but $\chi \neq 0$: ϕ will evolve (+ visa versa)
  ↳ source terms

Also: $$\mathcal{H} = \underbrace{\mathcal{H}_{free}(\phi)}_{\frac{1}{2}(\pi_\phi^2 + (\nabla\phi)^2 + m^2\phi^2)} + \mathcal{H}_{free}(\chi) + \underbrace{\lambda\phi^2\chi}_{\mathcal{H}_{int}}$$

Nb: all Classical ↓

Consider Case: $\chi = 0$

ϕ: Single point-like 'lump' of ϕ at origen

$\phi^2 \propto \delta(\vec{x})$

$\phi^2(\vec{x}) = \frac{A}{m_\phi} \delta(\vec{x})$

$A = 1$

$[\mathcal{L}] = [(\partial\phi)^2] = \frac{1}{L^2}\phi^2$

$[\mathcal{L}] = \frac{E}{L^3} \Rightarrow [\phi^2] = \frac{E}{L} = E^2 \quad (\hbar = c = 1)$

$[\delta] = \frac{1}{L^3} = E^3$

$[m] = E$

~~For simplicity~~:

E.O.M:
$$\boxed{\left(\partial_t^2 - \nabla^2 + m_\chi^2\right) \chi(t, \vec{x}) = -\frac{\lambda}{m_\phi} \delta(\vec{x})}$$

$\Rightarrow$ Has the form of a Green's function (classical convention)

For simplicity, assume χ evolving slowly: $\partial_t \chi \to 0$
(we'll return to full case in time)

$$\left[\nabla^2 - m_\chi^2\right] \chi(\vec{x}) = \frac{\lambda}{m_\phi} \delta(\vec{x})$$

Solve: Use Fourier trick:

$$\chi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{\chi}(\vec{k}), \quad \delta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}}$$

Plugging in
$$\int \frac{d^3k}{(2\pi)^3} (-k^2 - m^2) e^{i\vec{k}\cdot\vec{x}} \tilde{\chi}(k) = \frac{\lambda}{m_\phi} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}}$$

$$(k^2 + m_\chi^2)\tilde{\chi}(k) = -\frac{\lambda}{m_\phi}$$

$$\boxed{\tilde{\chi}(\vec{k}) = \frac{-\lambda}{m_\phi(k^2 + m_\chi^2)}}$$

Green's function (in our approx)
Somewhat analogous to Propagator: Next week!

Inverse Fourier :

$$\chi(\vec{x}) = -\frac{\lambda}{m_\phi}\int \frac{d^3k}{(2\pi)^3}\,\frac{e^{i\vec{k}\cdot\vec{x}}}{m(k^2+m_\chi^2)}$$

Can evaluate using Complex analysis. Won't do here.

$$= -\frac{\lambda}{m_\phi}\,\frac{e^{-m_\phi x}}{4\pi x} \qquad x = |\vec{x}|$$

$$\mathcal{H}_{int} = \lambda\phi^2\chi \quad : \quad V_{int} = \int d^3x\, \mathcal{H}_{int} \quad , \quad \phi^2 = \frac{1}{m_\phi}\delta(\vec{x})$$

$$\boxed{V_{int}(r) = -\frac{\lambda^2}{m_\phi^2}\,\frac{e^{-m_\phi |r|}}{4\pi r}}$$

* Position dependent Interaction Energy

* Creates an attractive force between to ϕ lumps (even if $\chi=0$)
  ↳ force 'mediated' by χ

* Interaction is long range + inverse square for $m_\chi = 0$
  Short-range for large m_χ

  cf: Electrodynamics vs. weak force

* ↑ known as Yukawa potential

We did this purely classically (as a motivation)

Do do properly in QFT, require some formalisms
- → distributions (ϕ, π, δ)
- → Green's functions
- → complex integrations

Distributions + Dirac delta

See Notes: 2.1

* Come across several objects which are not really functions, but distributions

Eg. Dirac delta "function"

→ For us: appeared as continuous analoge of Kronecker $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

→ $\delta(\vec{x}) \overset{?}{=} \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$ ~ fine as loose definition

$$\int_a^b f(x)\,\delta(x)\,dx = f(0) \qquad \begin{matrix} a<0<b \\ (\cancel{\quad}) \end{matrix}$$

δ: Not a function but a distribution, D

↳ linear functional : defined by its action on a function, f

$$D: f \longrightarrow \mathbb{R}$$

In general, e.g., $T_g[f] = \int g(x)\,f(x)\,dx$ — one possible example. In general could be any reasonable definition

$$\boxed{T_\delta[f] \equiv f(0)}$$

Delta $T_\delta[f] = \int \delta(x)\,f(x)\,dx$

* Not a function: $\delta(x) = 0$ for all $x \neq 0$ → has zero measure ⇒ $\int \delta = 0$
* δ: behaves like a fn under integration
* $T_\delta[f] = f(0)$ perfectly well defined

"Test functions", f, must be convergent on C^α_Ω

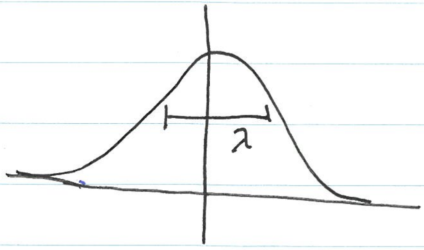
α: Differentiable α times

Ω: Subset of Reals

Representation of δ:

Consider Gaussians: $\quad u_\lambda(x) = \frac{1}{\sqrt{\pi}\,\lambda} e^{-x^2/\lambda^2}$

$\int u_\lambda(x)\, dx = 1 \quad$ for any λ



$$u_\lambda = \frac{1}{\lambda} u_1(x/\lambda)$$

$$T_{u_\lambda}[f] = \int dx \; \frac{1}{\lambda} u_1(x/\lambda)\, f(x) \qquad \text{let } y = x/\lambda$$

$$= \int dy \; u_1(y)\, f(y\lambda) \qquad \frac{dy}{dx} = \frac{1}{\lambda}$$

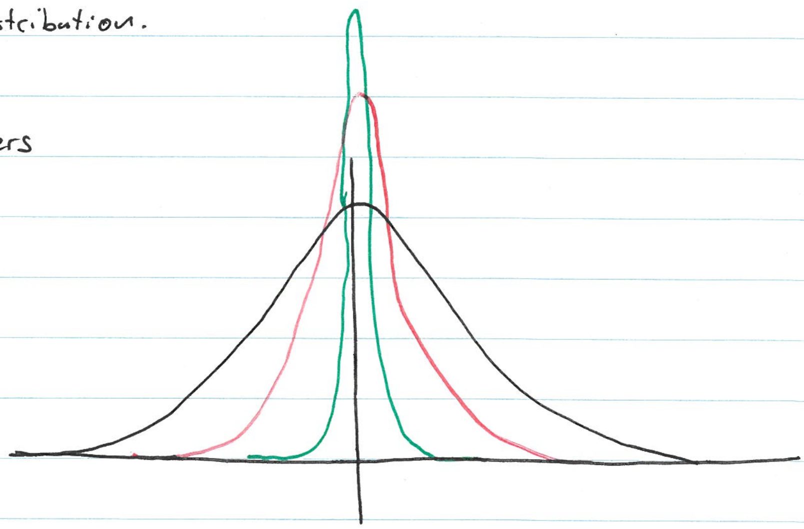
Take $\lambda \mapsto 0$

$$T_{u_0}[f] = \lim_{\lambda \to 0} \int dy \; u_1(y)\, f(\overbrace{y\lambda}^{0})$$

$$= f(0) \overbrace{\int dy \; u_1(y)}^{=1}$$

$\Rightarrow \quad u_\lambda$ with lim $\lambda \to 0$ is a representation of the Dirac delta distribution.

$\rightarrow$ Not unique - many others



* can be nice way to examine properties.

Simple Properties:

$$T_g[f] = \int g(x) f(x)\, dx$$

→ translate

$g(x) \to g(x-y)$: $\int g(x-y)\, f(x)\, dx$

Derivative: $\int dx \frac{\partial g}{\partial x} f(x) = - \int dx\, g(x) \frac{\partial f}{\partial x}$

via IBP

↳ $f \to 0$ at boundary

* Fourier transform

$$\tilde{\delta} = \int dx\, e^{ipx} \delta(x) = e^{ip0} = 1$$

Inverse ⇒ $\delta(x) = \int \frac{dP}{2\pi} e^{-ipx} \underbrace{\tilde{\delta}}_{= 1}$

* $\boxed{\delta(x) = \int \frac{dP}{2\pi} e^{-ipx}}$ Integral (Fourier) representation of δ

$\delta^{(3)}(\vec{x})$?

$$\int d^3x = \int dx\, dy\, dz$$

$$\Rightarrow \int d^{(3)}(x) \to \int \delta(x)\, \delta(y)\, d(z)$$

---

* $\delta(ax) = \frac{1}{a} \delta(x)$

Other 'distributions'

$$\delta$$

$$\theta(x) = $$ 

etc

Also: $\hat{\phi}$, $\tilde{\pi}$, $\tilde{a}$, a^{+} : operator distributions

$|\vec{p}\rangle$ – "particles", but not localised in space

$$\langle \vec{p} | \vec{p} \rangle = \langle 0 | a_p a_p^{+} | 0 \rangle = (2\pi)^3 \delta^{(3)}(0)$$

What about $\phi(x)$?

$$\langle 0 | \phi | 0 \rangle = 0 \qquad \text{ok}$$

Consider fluctuations : $\Delta \hat{o} = \hat{o} - \langle \hat{o} \rangle$

$$\sigma_o^2 = \langle \Delta o^2 \rangle$$
$$= \langle \hat{o}^2 \rangle - \langle a \rangle^2$$

$$\langle 0 | \phi^2 | 0 \rangle = ?$$
$$= \int \frac{d^3k \, d^3p}{(2\pi)^6 \, 2\sqrt{E_p E_k}} \langle 0 | \dots (a u + a^{+} u^*)(a^{+} u^* + a u) \dots | 0 \rangle$$
$$= \infty \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \quad \sim \quad \frac{k^2 \, dk}{\sqrt{k^2+m^2}} \rightarrow \infty$$

Useful relation we use alot:

$$\boxed{\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}}$$

x_i are roots of f
$f(x_i)=0$

With δ a distribution, proof is simple

$$\int \delta(f(x))\, g(x)\, dx = \sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} \delta(f(x))\, g(x)\, dx$$

non-zero only at each x_i

let $u = f(x)$, $x = f^{-1}(u)$, $\frac{du}{dx} = f'(x)$

in small enough region, inverse must exist.

$$= \sum_i \int_{f(x_i-\varepsilon)}^{f(x_i+\varepsilon)} \delta(u)\, g(f^{-1}(u))\, \frac{du}{f'(x)}$$

$$\int_a^b \delta(x) f(x) = f(0) \quad \text{for } b>a$$

$$\int_b^a \delta(x) f(x) = -f(0)$$

if $f(x_i+\varepsilon) < f(x_i-\varepsilon)$ then $f'(x)|_{x_i} < 0$
$\Rightarrow$ signs cancel

$$\boxed{= \sum_i \frac{g(x_i)}{|f'(x_i)|} \equiv \int \delta(f(x))\, g(x)\, dx}$$

$$\Rightarrow \quad \delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$$

Green's Functions

$$\underbrace{\mathcal{L}}_{\text{Linear differential operator (in } x)} u(x) = \overbrace{f(x)}^{\text{"Source" Term}} \quad : \quad \text{Solve for } u$$

(e.g. $\partial^2 + m^2$ from K.G)

$$\mathcal{L}\vec{u} = \vec{f} \quad \rightarrow \quad \vec{u} = \mathcal{L}^{-1}\vec{f} \quad ?$$

Associate a Green's function, G, to $\mathcal{L}$:

$$\boxed{\mathcal{L}\, G(x,x') = \delta(x-x')}$$

← defines G

why?

→ Assume G is known

→ multiply by Source $f(x')$ + integrate

$$\int dx' \, \mathcal{L}\, G(x,x')\, f(x') = \int f(x')\, \delta(x-x')\, dx'$$

$\mathcal{L}$ is linear, acts on x

$$\mathcal{L} \underbrace{\int dx' \, G(x,x')\, f(x')}_{u(x)!} = f(x)$$

By definition!

$$\boxed{u(x) = \int dx' \, G(x,x')\, f(x')}$$

Solve Inhomog. ODE by doing an integral!

General: case Some solution to $\mathcal{L} u_0 = 0$ ~~im~~ homog.

$$u = u_0 + u_*$$

$$\mathcal{L}u = f \quad \Rightarrow \quad \mathcal{L}u_* = f$$

Green's fn gives u_*

u_0: often removed by Boundary Conditions.

OK, but how to find G?

Fourier Transform!

Example:

$$L = \frac{\partial^2}{\partial x^2}$$

$$\frac{\partial^2}{\partial x^2} G(x,x') = \delta(x-x')$$

$$G(x,x') = \int \frac{dk}{2\pi} \tilde{G}(k) e^{+ik(x-x')}$$

$$\delta(x-x') = \int \frac{dk}{2\pi} e^{ik(x-x')}$$

$$\Rightarrow \int \frac{dk}{2\pi} \tilde{G}(k) \frac{\partial^2}{\partial x^2} e^{ik(x-x')} = \int \frac{dp}{2\pi} e^{ip(x-x')}$$

$$\int (ik)^2 \tilde{G}(k) e^{ik(x-x')} \frac{dk}{2\pi} = \int \frac{dp}{2\pi} e^{ip(x-x')}$$

$$\Rightarrow (ik)^2 \tilde{G}(k) = 1$$

$$\tilde{G}(k) = \frac{-1}{k^2}$$

$$G(x-x') = -\int \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{k^2}$$

Note:

1) Green's function not unique

Full solution requires boundary conditions

* Sometimes different conventions

e.g. $Lu = \pm i \delta$

Lorentz Invariant Normalisation

(see notes: 3.3
P&S : end of 2.3)
Tong: 2.4.1

$\langle 0 | 0 \rangle = 1$ great

Also: $\langle \vec{p} | \vec{q} \rangle \equiv \langle 0 | a_p a_q^\dagger | 0 \rangle = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$

Scalar ? No!

$$\underbrace{\int d^3p}_{\text{Not scalar}} \; \delta^{(3)}(\vec{p}) = \underbrace{1}_{\text{Scalar}}$$

$\Rightarrow \delta^{(3)}(p)$ is not scalar

Guess: dimensional analysis $[\delta^{(3)}] = \frac{1}{E^3}$ $\quad E \, \delta^{(3)}(p)$ is scalar

know d^4p is scalar

$$\int \frac{d^3p}{(2\pi)^3} \, g(\mathcal{E}_p, \vec{p}) \sim \int \frac{d^4p}{(2\pi)^4} \, g(p^\mu) \, (2\pi) \, \delta(p_\mu p^\mu - m^2) \, \underbrace{\Theta(p_0)}_{\text{ensures } p_0 > 0}$$

Use ugly relation

$$\boxed{\delta(f(x)) = \sum_{x_0} \frac{\delta(x - x_0)}{|f'(x_0)|}}$$

all zero's, $f(x_0) = 0$

$\rightarrow$ More on this next week.

$$\delta(p_0^2 - \vec{p}^{\,2} - m^2)\Big|_{p_0 > 0} = \frac{\delta(p_0 - \mathcal{E}_{\vec{p}})}{2 \mathcal{E}_{\vec{p}}}$$

zeros: $\pm \mathcal{E}_{\vec{p}}$
Θ: chooses +ve

$$\rightarrow \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\mathcal{E}_{\vec{p}}}}_{\text{Lorentz invariant } \vec{p} \text{ integral}} \; \underbrace{g(\mathcal{E}_p, \vec{p})}_{\text{May or may not be Lorentz invariant.}}$$

So, define: $\langle p | q \rangle = (2\pi)^3 \; \underbrace{2 E_{\vec{p}} \; \delta^{(3)}(\vec{p}-\vec{q})}_{\text{Lorentz Invar}}$

Re define: $\tilde{a}(\vec{p}) \longrightarrow \sqrt{2 E_{\vec{p}}}\; a_{\vec{p}}$

$$|p\rangle = \sqrt{2 E_{\vec{p}}}\; a^{+}_{p} |0\rangle$$

$$= \tilde{a}^{+}(\vec{p})\, |0\rangle$$

Common notation

$\overline{|p\rangle = \sqrt{2 E_p}\, |\vec{p}\rangle}$

Now:

$$[\phi(\vec{x}), \pi(\vec{y})] = i\, \delta^{(3)}(x-y)$$ same

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \underbrace{\frac{1}{2 E_{\vec{p}}}}_{\text{Before : } \frac{1}{\sqrt{2E}}} \left(\tilde{a}_p e^{i\vec{p}\cdot\vec{x}} + \tilde{a}^{+}_p e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\pi(\vec{x}) = \frac{-i}{2} \int \frac{d^3 p}{(2\pi)^3} \left(\tilde{a}_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - \tilde{a}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$[\tilde{a}_{\vec{p}}, \tilde{a}^{+}_{\vec{q}}] = 2 E_{\vec{p}}\, (2\pi)^3\, \delta(\vec{p}-\vec{q})$$

$$\hat{H} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2 E_{\vec{p}}} E_{\vec{p}}\, \tilde{a}^{+}_{\vec{p}} \tilde{a}_{\vec{p}}$$

Be cautious.

Notes: $\tilde{a}$ written just as a

P & S: uses 'old' a

Tong: defines $\hat{a}_{\vec{p}} = a(\vec{p})$

Sorry...

$$\phi_{(\vec{x})} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\mathcal{E}_p}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^+_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right) \qquad [a_p, a^+_q] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$$

$$\phi(\vec{x})\,|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\mathcal{E}_p}}\, e^{-i\vec{p}\cdot\vec{x}}\, |\vec{p}\rangle$$

$$= \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\mathcal{E}_p}\, e^{-i\vec{p}\cdot\vec{x}}\, |p\rangle}_{\text{Eigenstate of pos : } |x\rangle}$$

[Using Tong Notation
$|\vec{p}\rangle = a^+_p|0\rangle$
$|p\rangle = \sqrt{2\mathcal{E}_p}\, a^+_p|0\rangle$]

Check: $\langle 0|\,\phi(\vec{y})\,\phi(\vec{x})\,|0\rangle$ should be $\langle x|y\rangle = 0$

ϕ: 'creates' particle @ position $\vec{x}$

Check: $\langle x|p\rangle$ should be $\propto e^{i\vec{p}\cdot\vec{x}}$ for 'wave function' of state $|p\rangle$

$$\langle 0|\,\phi(x)\,|p\rangle = \langle 0| \left[\int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\mathcal{E}_k}} \left(a_k\, e^{i\vec{k}\cdot\vec{x}} + a^+_k\, e^{-i\vec{k}\cdot\vec{x}} \right) \right] \sqrt{2\mathcal{E}_p}\, a^+_p\, |0\rangle$$

$$a_k\, a^+_p = \left([a_k, a^+_p] + a^+_p\, a_k \right) |0\rangle$$

$$\Rightarrow \left((2\pi)^3\, \delta(\vec{k}-\vec{p}) + 0 \right) |0\rangle$$

$$\langle 0|\, a^+_k\, a^+_p\, |0\rangle = 0 \qquad \text{i.e. } \langle 0|2\rangle$$

$$\langle 0|\,\phi(x)\,|p\rangle = \langle 0| \int d^3k\ \delta(\vec{k}-\vec{p})\ e^{i\vec{k}\cdot\vec{x}}\ |0\rangle$$

$$= e^{i\vec{p}\cdot\vec{x}} \qquad \text{OK!}$$

B. M. Roberts

PHYS 4040: QFT – Lecture 6 – W3(B) – 2024

Complex Analysis

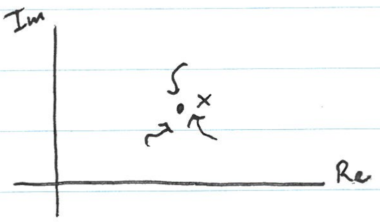
→ Some tools from complex analysis extremely useful

Analytic functions + functions non-analytic at set # of points

(Terminology: Analytic = Regular = holomorphic)

Analytic? Differentiable. Complex: slightly more complicated

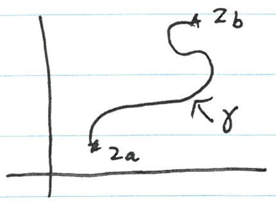
$$\lim_{\delta\to 0} \frac{f(x+\delta)-f(x)}{(x+\delta)-x}$$



Analytic: limit $x \to x+\delta$ the same

Contour

$$\int_{z_a,\,\gamma}^{z_b} dz\, f(z) = F(z_b) - F(z_a)$$



γ: $z(t)$ parametrised curve

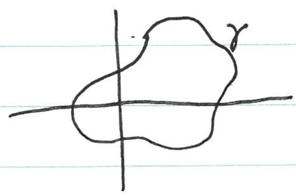
$$\int f(z(t))\, \frac{dz}{dt}\, dt$$

⇒ Cauchys Integral Theorem:

for analytic fn f:

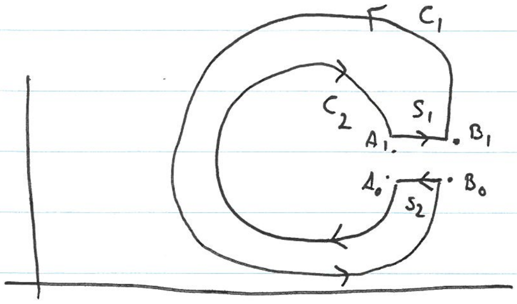
above w/ $z_b \to z_a$

$$\oint_\gamma f(z)\, dz = 0$$



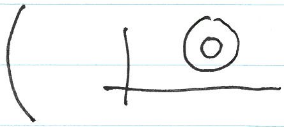
Important Result from this: Contour direction

Consider



$$\oint f = 0$$

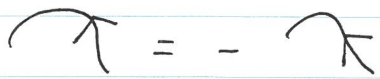
$$\Rightarrow \quad C_1 + C_2 + S_1 + S_2 = 0$$

let $A_0 \to A_1$, $B_0 \to B_1$ (

$$\oint_{C_1} f = 0 \quad \text{and} \quad \oint_{C_2} f = 0$$

$$\Rightarrow \quad \oint_{S_1} f = -\oint_{S_2} f$$

Same line, but opposite direction

$\Rightarrow$ Defines direction of Contour : 

Cauchy's Integral formula

for analytic f

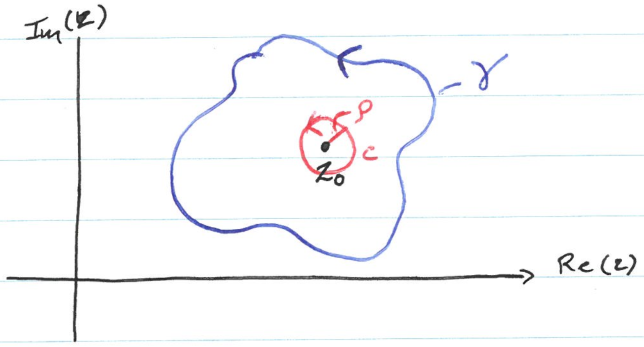
$$\boxed{\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-z_0} = \begin{cases} 0 & \text{if } z_0 \text{ outside } \gamma \\ f(z_0) & \text{if } z_0 \text{ inside } \gamma \end{cases}}$$

* If $f(z)$ is analytic, $\dfrac{f(z)}{z-z_0}$ is not

* Non-analytic at $z=z_0$: called a pole (of $\frac{f}{z-z_0}$, not of f)

Proove it :

Draw small circle, C, around z_0, radius ρ



Consider

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz - f(z_0)\int_C \frac{1}{z-z_0}\,dz \qquad (1)$$

In between γ and c, $\dfrac{f(z)}{z-z_0}$ is analytic

$\Rightarrow$ can deform the path

$$\Rightarrow \quad \oint_\gamma \frac{f(z)}{z-z_0}\,dz = \oint_c \frac{f(z)}{z-z_0}\,dz \qquad (2)$$

Combining, we have: (① and ②)

$$\oint_\gamma \frac{f(z)}{z-z_0}\,dz = \overbrace{f(z_0)\underbrace{\oint_C \frac{1}{z-z_0}\,dz}_{\text{Just a number, will evaluate soon.}}}^{C_1} + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

(A) $\displaystyle\oint_C \frac{1}{z-z_0}\,dz$ along circludr path c, $\rho = |z-z_0|$

let $z = z(\theta) = z_0 + \rho e^{i\theta}$ $\quad \theta: 0 \to 2\pi$

$dz = i\rho e^{i\theta}\,d\theta$

$$= \int_0^{2\pi} \frac{1}{z_0 + \rho e^{i\theta} - z_0}\, i\rho e^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} d\theta = 2\pi i$$

(B) $\displaystyle\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz$

To evaluate, note:

* $|f(z) - f(z_0)| \leq \varepsilon \to 0$ as $\rho \to 0$
* $|z - z_0| = \rho$ along c

Also: $$\left|\int_\gamma g(z)\,dz\right| \leq |\underbrace{\text{Max}(g)}_{\text{Max magnitude along path}}| \times \underbrace{L(\gamma)}_{\text{Path length of } \gamma}$$

Combining:

$$\left| \oint \frac{f(z)-f(z_0)}{z-z_0}\, dz \right| < \overbrace{\frac{\varepsilon}{\rho}}^{M} \cdot \overbrace{2\pi\rho}^{L}$$

$$< 2\pi\varepsilon$$

ε can be arbitrarily small, so $\to 0$

$$\Rightarrow \oint_{\gamma} \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0) \qquad \text{for } z_0 \text{ in } \gamma$$

Cauchy's Residue theorem:

$f(z)$ is analytic within γ
except for finite # of pointz: z_i (poles)

$$\boxed{\oint_{\gamma} f(z)\, dz = 2\pi i \sum_i \text{Residues}\left[f(z)\right]\Big|_{z=z_i}}$$

First order poles: $f(z) = \dfrac{g(z)}{z-z_i}$: - f has 1^{st} order pole at z_i
- $g(z)$ is analytic
- $g(z_i)$ is residue

n^{th} order pole $f(z) = \dfrac{g(z)}{(z-z_i)^n}$ for g analytic

n^{th} order pole

$$\text{Res}(f, z_i) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \overbrace{f(z)\cdot(z-z_i)^n}^{g} \Big|_{z_i = z_i}$$

Not proven: proove by taking derivatives of Cauey Integral Formula.

Why is it useful?

$$\boxed{\oint_\gamma f(z)\,dz = 2\pi i \sum_i \mathrm{Res}(f, z_i)}$$

* Makes certain contour integrals v. easy

* Can often cast non-contour integrals into form ↑ by extending into complex plane.

Example:

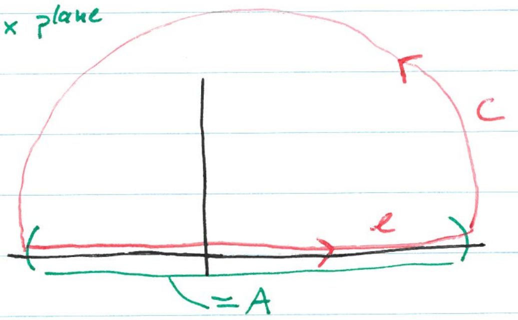
$$A = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

– Real x, real line only

Extend to complex plane

Now $$\oint_\gamma \frac{1}{1+z^2}\,dz$$

*



* Not in general $= A$!

$$= \overbrace{\int_e \frac{1}{z^2+1}\,dz}^{A} + \underbrace{\int_c \frac{1}{z^2+1}\,dz}_{?}$$

$$z_c = \rho e^{i\theta} \quad \text{with} \quad \rho \to 0, \quad \theta: 0 \to \pi$$

$$dz = i\rho e^{i\theta}\,d\theta$$

$$\int_c \frac{1}{z^2+1}\,dz = \int_0^\pi \frac{i\rho e^{i\theta}\,d\theta}{\rho^2 e^{2i\theta}+1}\Bigg|_{\rho\to\infty} \longrightarrow 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \oint_\gamma \frac{1}{z^2+1}\,dz$$

Big circle lemma

$$\oint = \int_{\mathbb{R}} + \int_{C(\rho)}$$

The integral over big arc, c, for $\rho \to \infty$ Vanishishes

So long as:

(Jordan's Lemma)

$$\lim_{|z|\to\infty} z\, f(z) = 0$$

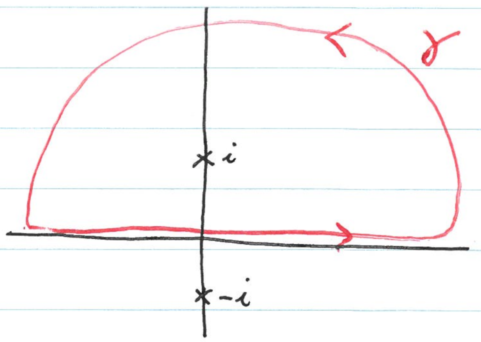
(you can be a little more precise if Required)

roughly Since $\left| \int_c f(z)\, dz \right| < \text{Max}(f)_c \cdot \pi\rho$ ~~$f\ldots$ $\rho\ldots$~~

Return to our problem

$$\oint_\gamma \frac{1}{z^2+1}\, dz = \underbrace{\int_{-\infty}^{\infty}}_{\text{Just along } \mathbb{R},\ x = \mathrm{Re}(z)} \frac{1}{x^2+1}\, dx + \underbrace{\overbrace{\int_C \frac{1}{z^2+1}\, dz}^{\text{Just along arc}}}_{\simeq 0}$$

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} \qquad \text{poles at } \pm i$$

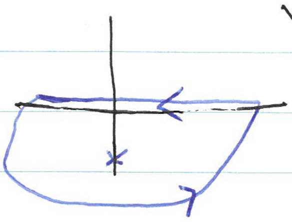


only $+i$ is inside γ

$$\text{Res}: \left.\frac{1}{(z+i)}\right|_{z=i}$$

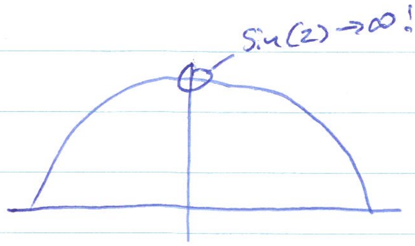
$$= \frac{1}{2i}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{x^2+1} = \frac{2\pi i}{2i} = \pi$$

(Q what if we chose other contour ? )

Another example:

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx$$



$$\oint \frac{\sin(z)}{z}\, dz = \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx + \underbrace{\int_c \frac{\sin(z)}{z}\, dz}_{\text{Zero?}}$$

Zero? $\sin(z)$ w/ $z \to i\infty$?

No!

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

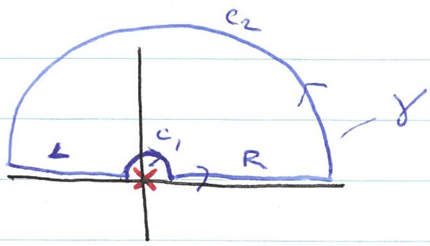
$$\oint_\gamma \frac{e^{iz} - e^{-iz}}{2iz} \quad \Big| \quad \underbrace{\int_{-\infty}^{\infty} \frac{e^{ix}}{2ix}\, dx}_{} - \underbrace{\int_{-\infty}^{\infty} \frac{e^{-ix}}{2ix}\, dx}_{}$$

$\to 0$ as $x \to i\infty$
$\Rightarrow$ upper semicircle

$\to 0$ as $x \to -i\infty$
lower semicircle

OR: $x \to y = -x$

$$= \int_{-\infty}^{\infty} \frac{e^{ix}}{ix}$$



pole at $x = 0$

$$\underbrace{\int_\gamma}_{\substack{\| \\ 0}} = \underbrace{\int_L + \int_R}_{\text{What we need}} + \int_{c_1} + \underbrace{\int_{c_2}}_{\substack{\| \\ 0}}$$

No poles in γ

$\| \; 0$ By Jordan

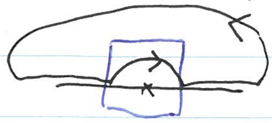
Radius of $c_1 \to 0$

So that $L + R = \mathbb{R}$

$$\underbrace{\int_{C+R} = \int_{-\infty}^{\infty} = -\int_{C_1}}$$

$$\int_{C_1} \frac{e^{iz}}{iz}\,dz \qquad z = \rho e^{i\theta} \qquad \theta = 0 \to \pi \qquad \rho \to 0$$

(notice our contour goes backwards!)



$$= \int_{\pi}^{0} \frac{\exp(\rho i e^{i\rho\theta})}{\cancel{i\rho}\,\cancel{e^{i\rho\theta}}}\,\underbrace{\cancel{i\rho}\,\cancel{e^{i\theta}}\,d\theta}_{dz}$$

$$= \int_{\pi}^{0} e^{\rho i e^{i\rho\theta}}\,d\theta \quad \Big|\ \lim \rho \to 0$$

$$= -\int_{0}^{\pi} 1 \, d\theta$$

$$= -\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$$

Trick for spherical Integrals

$$\int d^3\vec{x}\, f(\vec{x}) = \int_{-\infty}^{\infty} x^2\,dx \underbrace{\int_{0}^{\pi} \sin\theta\, d\theta}_{} \int_{0}^{2\pi} d\phi \quad f(x,\theta,\phi)$$

let $\gamma = \cos\theta$, $d\gamma = -\sin\theta\, d\theta$

$$\int_{0}^{\pi} f \sin\theta\, d\theta = \int_{1}^{-1} f \frac{\sin\theta}{-\sin\theta}\,d\gamma = \int_{-1}^{1} f\, d(\underbrace{\cos\theta}_{\gamma})$$

Causality

Back to Heisenberg picture

$$\phi(x) = \phi(x^\mu)$$

Amplitued to propagate from $y \to x$:

$$\langle 0|\phi(x)\,\phi(y)|0\rangle = \langle 0| \left[\int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\mathcal{E}_k}}\frac{1}{\sqrt{2\mathcal{E}_p}} \times \right.$$

$$\left(a_{\vec{k}}e^{-ik\cdot x} + a^+_{\vec{k}}e^{ik\cdot x}\right)\left(a_{\vec{p}}e^{-ip\cdot y} + a^+_{\vec{p}}\, e^{ip\cdot y}\right) |0\rangle$$

only cross term $\propto$ $a_k a^+_p$ survives

$$\langle 0|a_k a^+_p|0\rangle = (2\pi)^3\,\delta^{(3)}(\vec{k}-\vec{p})$$

$$\underbrace{\langle 0|\phi(x)\phi(y)|0\rangle}_{\equiv D(x-y)} = \int \frac{d^3k}{(2\pi)^3}\, d^3p\, \frac{1}{2\sqrt{\mathcal{E}_k\mathcal{E}_p}}\, e^{(-ik\cdot x + ip\cdot y)}\, \delta^{(3)}(\vec{k}-\vec{p})$$

$$\Rightarrow \quad D(x-y) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\mathcal{E}_k}\, e^{-ik\cdot(x-y)}$$

Let's evaluate for space-like separation, e.g. $x^0 - y^0 = 0$, $\vec{x}-\vec{y} = \vec{r}$

$$D(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\mathcal{E}_k}\, e^{+i\vec{k}\cdot\vec{r}}$$

$$= 2\pi \int_0^\infty \frac{k^2\,dk}{(2\pi)^3}\int_{-1}^{1} d\gamma\, \frac{e^{+ikr\gamma}}{2\sqrt{k^2+m^2}}$$

Now, $k = |\vec{k}|$

$$D = \frac{1}{2(2\pi)^2}\int_0^{\infty}\frac{k^2}{\sqrt{k^2+m^2}}\,dk\left[\frac{1}{ikr}\left(e^{ikr} - \underbrace{e^{-ikr}}_{k\to -k \quad \int_0^{\infty}\to\int_{-\infty}^{0}}\right)\right]$$

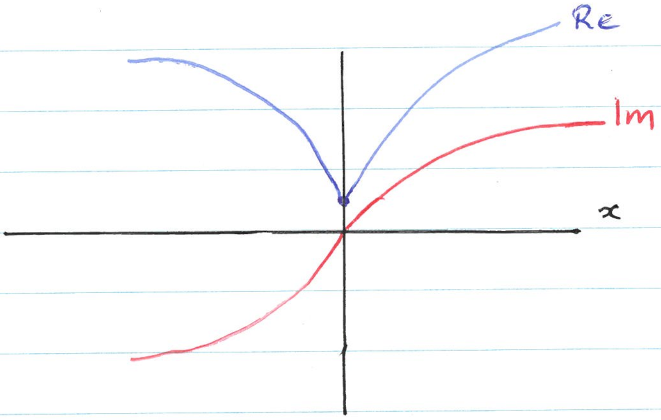
$$= \frac{1}{8\pi^2 ri}\int_{-\infty}^{\infty} dk\,\frac{k}{\underbrace{\sqrt{k^2+m^2}}}\,e^{ikr}$$

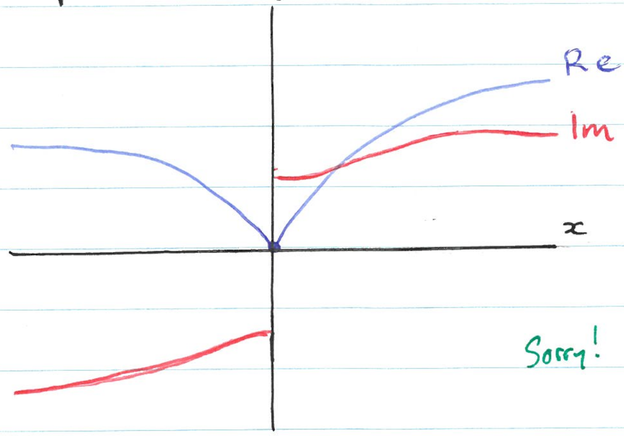
oof. poles at $k = \pm im$, not simple poles

Worse: discontinuous for all $k = \pm i\rho$ for $\rho > m$

$\left(\frac{k}{m} = x + i \pm i\varepsilon\right)$

$\sqrt{[x+i(1-\varepsilon)]^2+1}$ $\qquad\qquad$ $\sqrt{[x+i(1+\varepsilon)]^2+1}$



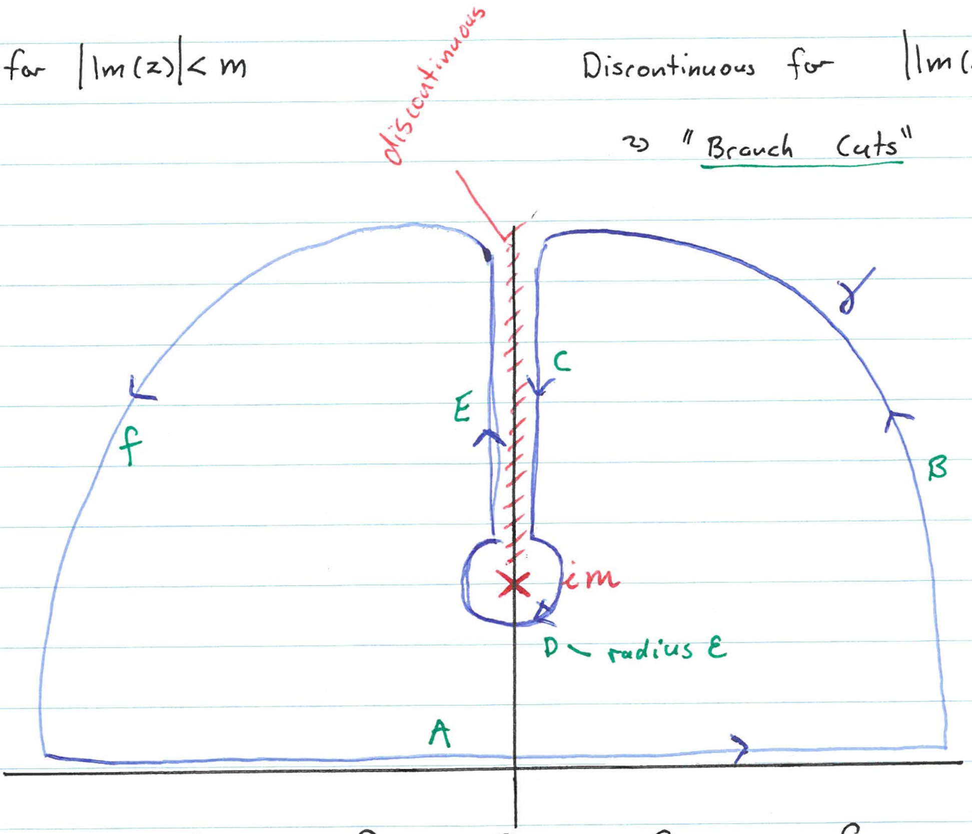


Sorry!

Fine for $|\mathrm{Im}(z)| < m$ $\qquad\qquad$ Discontinuous for $|\mathrm{Im}(z)| > m$

$\Rightarrow$ "Branch Cuts"

$\frac{k}{\sqrt{k^2+m^2}}$



Avoid all Non-Analytic pts!

$$\oint_\gamma = 0 = \int_A + \int_B + \int_C + \int_D + \int_E + \int_F$$

Zero (Jordan)*

$$0 = \underbrace{A}_{\text{what we want}} + \overbrace{(B+F)}^{\text{zero (Jordan)}^*} + \underbrace{D}_{\text{zero } \epsilon\to 0} + \underbrace{C + E}_{C=E}$$

* Actually, these don't go to zero in general. But, we'll take $r \to \infty$ also, so then they do.

- have $C = s_1 s_2 E$

$s_1 = -1$ from contour direction

$s_2 = -1$ from branch cut (change in $\sqrt{\ }$ sign)

$$A = -2C \qquad (C \text{ is } \infty \to m)$$

$$\Rightarrow \quad A = \frac{1}{4\pi^2 r i} \int_{im}^{\infty} \frac{k}{\sqrt{k^2+m^2}} e^{ikr} \, dk$$

$$= \frac{1}{4\pi^2 r} \int_{m}^{\infty} \frac{\rho \, e^{-\rho r}}{\sqrt{\rho^2 - m^2}} \, d\rho$$

For $r \gg \rho \qquad \sim \frac{1}{4\pi^2 r^2} e^{-mr}$

Exponentially suppressed, but not zero!

(NB: Sorry! I didn't remember how long that calc. was until I did it!)

OK, Step back.

Had: Equal-time comutation relations

$$[\phi(\vec{x},t)\,,\ \pi(\vec{y},t)] = i\,\delta^{(3)}(\vec{x}-\vec{y})$$

Need to think about measurements:

To protect causality, measurements outside the light-cone must not impact each other

Now 4 vectors, x^μ, y^μ, $x^0 \neq y^0$

$$\Rightarrow \Delta(x-y) \equiv [\phi(x)\,,\ \phi(y)] = 0 \quad \text{for space-like separated } x, y$$

? Is it?

We calculated already

$$\langle 0|\,\phi(x)\,\phi(y)\,|0\rangle = D(x-y)$$

note $[\phi,\phi]$ is a number

$$\Rightarrow \quad [\phi,\phi] = \langle 0|\,[\phi,\phi]\,|0\rangle$$

$$\Rightarrow \Delta = [\phi(x)\,,\ \phi(y)] = D(x-y) - D(y-x)$$

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\mathcal{E}_{\vec{p}}}\left(e^{-ip\cdot(x-y)} - e^{ip(x-y)}\right)$$

can also just directly calculate

- Non-zero fore time-like sep $\sim\left(e^{-i\mathcal{E}\Delta t} - e^{i\mathcal{E}\Delta t}\right)$
- Space-like? $x-y = -(\vec{x}-\vec{y}) < 0$ (simplest, eaqual time)

  Clearly zero beause [ ]

- In general: Δ is Lorentz invarient $\quad d^3p\,\frac{1}{\mathcal{E}}$, $\quad p\cdot x$
  - $\Rightarrow$ only depends on $(x-y)^2$
  - $\Rightarrow$ Can swap sign of $\vec{p}$, integration variable
  - $\Rightarrow$ Lorentz transforms (orthochronous): always a frame where space like poits, x, y, have same t.
  - $\Rightarrow$ Vanishes for any $(x-y)^2 < 0$ [space-like]