25/7/24 B.M. Roberts

PHYS 4040: QFT 2024 Lecture 1 W1(A)

Notes: 1.1–1.3
Tong: ~1

So far:
- Classical Mechanics
- Relativity
- Classic Fields (e.g. Maxwell)
- Quantum Mechanics (Non-relativistic)
- Perhaps: "Relativisitic QM" (Dirac) ..

QFT: unifies Relativistic field theory with QM

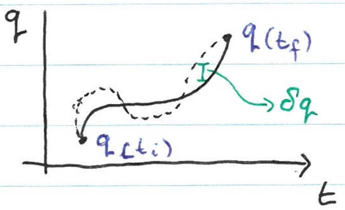
What about RQM? - Practically, works OK in ~~many~~ cases
- But, fundamentally fails

Screaming Overview: Classical Mechanics

N particles: $2 \times 3N$ coords q_i, $\dot{q}_i$

(can be $\lesssim 6N$)

Determanistic paths:



Which path does nature choose?
"Best" (i.e., optimal) wrt some measure: S - "Action"

$\Rightarrow \quad \delta S = 0$ Principle of Stationary Action

S: not 'normal' function: depends on entire path

$$\Rightarrow \quad S = \int L(q, \dot{q})\, dt$$

Local function of dynamical variables

$$\delta S = \int \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] dt = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}}$$

E.L.

→ Said nothing of form of L

* L: local

* Free : no explicit q or t dependance (Homog)

: no explicit direction

Free: $L_0 = L_0(v^2)$

+ External Influence: $L = L_0(v^2) + \overbrace{L(q)}^{-V(x)}$

Breaks exact Homog.

Classical:

p: generator of spatial translation

H: generator of time translations

Canonical / conjugate momentum: $p_i = \frac{\partial L}{\partial \dot{q}_i}$

Hamiltonian $H = \sum_i p_i \dot{q}_i - L$ | use p ↑ replace $\dot{q} \to p$

Hamilton's e.o.m: Hamilton Formulation

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

See C.M 1.5

From

$\delta H = p_i \delta \dot{q}_i + \delta p_i \dot{q}_i - \delta L$

$= \delta p_i \dot{q}_i - \dot{p}_i \delta q_i$

Also $\delta H = \frac{\partial H}{\partial q} \delta q + \frac{\partial H}{\partial p} \delta p$

Poisson bracket

$$\{f, g\}_{PB} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

⇒ derivatives: $\dot{f} = \frac{\partial f}{\partial t} + \{f, H\}$

$$\dot{q} = \{q, H\} \quad , \quad \dot{p} = \{p, H\}$$

$$\{p_i, p_j\} = 0 \quad , \quad \{q_i, q_j\} = 0 \quad , \quad \{q_i, p_j\} = \delta_{ij}$$

Q: Difference between CM & QM?

QM : probabilistic

state vectors $|\psi\rangle$ live at Hilbert's place $(|\psi\rangle \in \mathcal{H})$

$$\hat{H}|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle$$

$\psi(t, \vec{x})$
t: parameter
x: observable, e.val of operator $\hat{x}$

observables: operators $\quad q, p \mapsto \hat{q}, \hat{p}$

$\langle\psi|\hat{o}|\psi\rangle$
↳ probability amplitude

$$P(f) = |\langle\psi|\hat{f}|\psi\rangle|^2$$ prob of finding 'f' if measured made on state ψ

$$\{q_i, p_j\}_{PB} = \delta_{ij}$$

$$\rightarrow \quad [\hat{q}_i, \hat{p}_j] = i\hbar\,\delta_{ij} \quad \longrightarrow \text{Canonical Commutation Relation.}$$

$[A,B] = AB - BA$
$\{A,B\}_{PB} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$
$\{A,B\} = AB + BA$
↳ anti-commutator

or $\quad [\hat{f}, \hat{g}] = i\hbar\{f, g\}_{PB}$

Schrödinger picture : states evolve $\quad |\psi(t)\rangle$
operators don't $\quad \hat{x}, \hat{p}$

Heisenberg: operators evolve $\quad \hat{x}(t), \hat{p}(t)$
states don't $\quad |\psi\rangle$

Time evolution operator $\mathcal{U}(t, \delta t)$: $\hat{\mathcal{U}}|\psi(t)\rangle = |\psi(t+\delta t)\rangle$

Consider $\mathcal{U}(t, \delta t) \approx 1 + \delta t\, G$ for some G

$$(1 + \delta t\, \hat{G})|\psi(t)\rangle = |\psi(t+\delta t)\rangle$$

(up to factor)
Generator of time translations!
Just as in classical

$$\underbrace{\frac{|\psi(t+\delta t)\rangle - |\psi(t)\rangle}{\delta t}}_{\frac{\partial\psi}{\partial t}} = G\,|\psi\rangle$$

$$\Rightarrow \quad G = \frac{-i}{\hbar} H$$

$$U(t_0, \delta t) = 1 - \frac{i}{\hbar}\delta t \hat{H}$$

$$U(t_0, \Delta t) = \left(1 - \frac{i}{\hbar}\delta t \hat{H}\right)^N$$

$\Delta T = N\delta t$

$N \to \infty$

$t_0 \to 0$ wlog

$$\to e^{-iHt/\hbar}$$

Unitary: $U^{+}U = 1$

$\langle a \rangle_{(t)} = \langle \psi(t) | \hat{a} | \psi(t) \rangle_S$

$= \langle \psi(0) | U^{+} \hat{a} U | \psi(0) \rangle$

$\to$ Heis

$\langle a \rangle(t) \to \langle a(t) \rangle$

$\underset{\uparrow \text{ Heis}}{\hat{a}(t)} = U^{+} \underset{\nwarrow \text{ Schrö}}{\hat{a}} U$

Heisenberg:

$$\hat{f}(t) = U^{+} \hat{f}_{\text{Schrö}} U$$

H.E.O.M :

$$\frac{d}{dt}\hat{f} = \frac{i}{\hbar}[H, \hat{f}] + \underbrace{\frac{\partial f}{\partial t}}_{\text{usually } 0}$$

RQM: Relativistic Quantum Mechanics

Time + Space: treated differently

↳ Very stark for multi-particle states

$$\psi = \psi(t, x_1, x_2, \ldots)$$

Schrö

$E = \frac{p^2}{2m} + V$ Non-rel

$\left(i\partial_t + \frac{\nabla^2}{2m} - V\right)\psi = 0$

Deeper issues :

attempt $E^2 = p^2c^2 + m^2c^4$

$\to E = \sqrt{p^2 + m^2}$

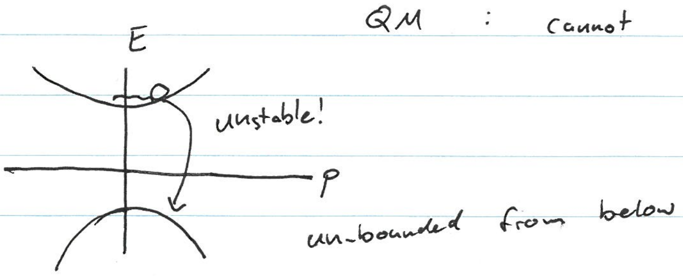
$H \overset{?}{=} \sqrt{\hat{p}^2 + m^2}$ KG or Dirac

$(\partial^2 + m^2)\psi = 0$

$\Rightarrow (E^2 - p^2 - m^2)\psi = 0$

$E = \pm\sqrt{p^2 + m^2}$ Classically: just discard −ve E states

QM : cannot − required for completeness



Big problem:

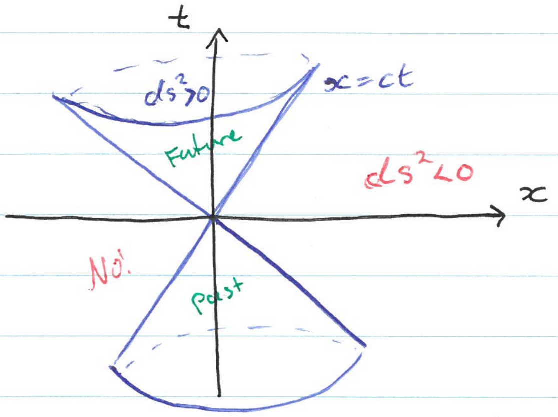
→ No consistant probabilistic interpretation

2) Causality is violated

Points x^μ, y^μ causally connected

only if $dx^2 \geq 0$

$$(x^\mu - y^\mu)^2 = (x^0 - y^0)^2 - |\vec{x} - \vec{y}|^2 \geq 0$$



[See, e.g. P&S 2.1]

In QM?

$$A = \langle \vec{y} | e^{-iHt} | \vec{x} \rangle$$

Non-rel : $H \simeq \frac{p^2}{2m}$ $\quad A \sim e^{im(\Delta\vec{x})^2/2t}$

rel : $H = \sqrt{p^2 + m^2}$ $\quad A \sim e^{-m\sqrt{\Delta x^2 - t^2}}$

Non-Rel	Rel
$\int \frac{d^3p}{(2\pi)^3} \langle \vec{y} \vert e^{-it\frac{\vec{p}^2}{2m}} \vert \vec{p} \rangle \langle \vec{p} \vert \vec{x} \rangle$	$\int \frac{d^3p}{(2\pi)^3} e^{-it\sqrt{p^2+m^2}} e^{i\vec{p}\cdot\Delta\vec{x}}$
$\simeq \int e^{-\frac{p^2}{2m}t} e^{i\vec{p}\cdot(\Delta\vec{x})} \frac{d^3p}{(2\pi)^3}$	
$= \left(\frac{m}{2\pi i t}\right)^{3/2} e^{im(\Delta x)^2/2t}$	

Mainly: No way to deal w/ variable # of particles!

$E = mc^2 \Rightarrow$ particles created + destroyed.

QFT Solves all these problems.

QM: $q, p \longrightarrow \hat{q}, \hat{p}$ w/ $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$

$\hat{p}^\mu = i\hbar\partial^\mu$

QFT: $\psi \longrightarrow \hat{\psi}$

"Firts" Quantisation

"Second" Quantisation

(old terminology)

Classical field theory:

$\underline{q}(t) \longrightarrow \underline{\phi}(\vec{x}, t)$

just a label

Dynamical Variable

→ Continuous fields

i.e. ∞ spatial degrees of freedom

$\hat{\phi}$	Scalar field	(e.g. Higg's)	also: simplest	spin 0
$\hat{\psi}$	Spinor fied	(Dirac, Fermions)		spin $1/2$
$\hat{F}_{\mu\nu}$	Photon	Vector field		Spin 1

etc

SR + tensor notation

$$ds^2 = \quad = \quad dt^2 - dx^2 \quad = \quad dx_\mu dx^\mu \qquad \eta = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$a_\mu b^\mu = \sum_{\mu=0}^{3} a_\mu b^\mu$$

$$= a^0 b^0 - \vec{a} \cdot \vec{b}$$

$$\boxed{\begin{matrix} \mu : 0-3 & \\ & \text{convention} \\ i : 1-3 & \end{matrix}}$$

$$a_\mu = \eta_{\mu\nu} a^\nu \qquad \eta_{\mu\nu} = \eta^{\mu\nu}$$

$$\eta_\mu^{\ \nu} = \delta_\mu^{\ \nu} = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

Lorentz: $x^{\mu'} = \Lambda^\mu{}_\nu x^\nu$ contra variant

$x'_\mu = \Lambda_\mu{}^\nu x_\nu$ co variant

$$\Lambda_\mu{}^\nu = (\Lambda^{-1})^\mu{}_\nu$$

3 boosts + 3 Rotations

x - boost

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma = \left(\sqrt{1 - v^2/c^2}\right)^{-1}$$

~~x~~ z - Rotation:

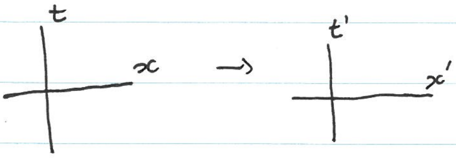
$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & +\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ASIDE

Relativity

- Laws of physics same in all ref frames
- c const. in " "

Link coords



?

Light beam: $\vec{x}_1$ to $\vec{x}_2$

$$(\Delta x)^2 = D^2$$
$$(c\,\Delta t)^2 = D^2$$

$$\Rightarrow (c\Delta t)^2 - (\Delta x)^2 = 0 = (c\,\Delta t')^2 - (\Delta x')^2$$ → Light-like separated

general: $\Rightarrow ds^2 = c\Delta t^2 - \Delta x^2$ Invariant Interval

Aside:

"Proof" Three frames: K_1, K_2, K_3

$\vec{V}_2$, $\vec{V}_3$

$$ds_2^2 = f(v_2^2)\, ds_1^2 \qquad ds_3^2 = f(v_3^2)\, ds_1^2$$

also $$ds_3^2 = f(|v_3 - v_2|^2)\, ds_2^2$$

$|\vec{V}_3 - \vec{V}_2|$ depends on $|V_3|$, $|V_2|$, and $\cos\theta$

no θ!

$$\Rightarrow f(v_{32}^2) = \frac{f(v_3^2)}{f(v_2^2)}$$

$$\Rightarrow f = 1$$

$$\partial_\mu \equiv \frac{d}{\partial x^\mu}$$ <u>co</u> variant!

(transforms as x_μ, not x^μ)

$$\partial_\mu = (\partial_0, +\vec{\nabla})$$

$$\partial^\mu = (\partial_0, -\vec{\nabla})$$ ['contra' deriv]

$$\partial^2 = \partial_\mu \partial^\mu = \partial_t^2 - \nabla^2$$ Invariant

Higher rank : $T^{\mu\nu}$

$$T'^{\mu\nu'} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}$$ Rank 2

+ generalise!

(e.g. consider $C^{\mu\nu} = A^\mu B^\nu$)

Note:
Careful w/ Notation

$$M_{\mu\nu} A_\nu = M \cdot A$$
$$M_{\mu\nu} A_\mu = M^T \cdot A$$

Tensor (rank N) – N uncontracted indices
– transforms w/ N Λ's

Scalar – invariant
– all indices contracted

nb: A^3 (3-comp of A^μ) <u>Not</u> scalar

dt? Not scalar
$dV = d^3x$? No
d^4x? <u>yes</u>
etc.

$$\partial_\mu \partial^\mu \quad A^\mu A_\mu, \quad A^\mu B_\mu \quad F^{\mu\nu} F_{\mu\nu} \quad \text{etc.}$$

Common: $x = x^\mu = (x_0, \vec{x})$ less: $\hat{x} = x_\mu$

$x^2 = x_\mu x^\mu$

$a \cdot b = a_\mu b^\mu$

Consider Lorentz transform

→ infinitesimal

'Generator' of L.T

$$\Lambda = 1 + \varepsilon \lambda$$

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \varepsilon \lambda^\mu{}_\nu$$

nb: $T^\mu{}_\nu \neq T_\nu{}^\mu$ in general. Be careful!

Continuous Transform: repeated infinitesimal

$x^T \eta y = x^{T'} \eta y'$

$\Rightarrow \Lambda^T \cdot \eta \Lambda = \eta$

Λ must leave η unchanged

$\Rightarrow$ gives condition on λ

$$\Rightarrow \quad \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta} = \eta^{\mu\nu}$$

$$= (\delta^\mu_\alpha + \varepsilon \lambda^\mu{}_\alpha)(\delta^\nu_\beta + \varepsilon \lambda^\nu{}_\beta)\eta^{\alpha\beta}$$

$$= \eta^{\mu\nu} + \varepsilon(\lambda^\nu{}_\beta \eta^{\mu\beta} + \lambda^\mu{}_\alpha \eta^{\alpha\nu}) + \varepsilon^2 \ldots$$

$$= \eta^{\nu\nu} + \varepsilon \underbrace{(\lambda^{\nu\mu} + \lambda^{\mu\nu})}_{=0}$$

$\Rightarrow \quad \lambda^{\mu\nu} = -\lambda^{\nu\mu}$ anti-symmetric

$\Rightarrow$ 6 indep. components

$$\begin{pmatrix} 0 & \boxed{a} & 2 & 3 \\ \boxed{-a} & 0 & 4 & 5 \\ & & 0 & 6 \\ & & & 0 \end{pmatrix}$$

→ 3 boosts + 3 rotations!

(Aside)

units : $\hbar = c = 1$, + Heaviside-Lorentz
(No 4π in Maxwell Eq.)
$\varepsilon = \mu = 1$

$$[E] = [M] = \frac{1}{[L]} = \frac{1}{[T]}$$

* dimensional analysis : check code

hardly matters, but

$$\hbar c \approx 197 \text{ MeV} \cdot \text{fm}$$

Λ ~~is~~ forms a group

→ Identity $\delta_\mu^{\ \nu} = \eta_\mu^{\ \nu}$

→ Each Λ has Λ^{-1} $(= \eta \Lambda^T \eta)$

→ given Λ_1, Λ_2 : $\Lambda_3 = \Lambda_1 \Lambda_2$ is Lorentz
(closed)

(Not abelian $(\Lambda_1 \Lambda_2 \neq \Lambda_2 \Lambda_1)$

→ Associative $\Lambda_1 (\Lambda_2 \Lambda_3) = (\Lambda_1 \Lambda_2) \Lambda_3$

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3 examples : Λ : $\Lambda^T \eta \Lambda = \eta$ ⇒ $\det(\Lambda) = \pm 1$

1) Identity — $\det(\Lambda) = +1$ proper

2) $\begin{cases} T & \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \text{ Time reversal} \\ P & \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \text{ Parity / space reflection} \end{cases}$ — $\det(\Lambda) = -1$ Improper

3) Boosts, Rotation — $\det(\Lambda) = 1$ Proper

Lorentz → Proper Lorentz

More general algebraic structure

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\mu + \lambda^\mu{}_\nu + \dots$$

$$= \delta^\mu{}_\nu + \frac{i}{2}\lambda_{\alpha\beta}[J^{\alpha\beta}]^\mu{}_\nu$$

$$\Lambda = e^\lambda \equiv e^{-\frac{i}{2}\lambda_{\alpha\beta}J^{\alpha\beta}}$$

$J^{\alpha\beta} = -J^{\beta\alpha}$: 6 4×4 matrices

$\Rightarrow$ <u>Generators</u> of Λ

By definition, $\frac{i}{2}\lambda_{\alpha\beta}[J^{\alpha\beta}]^\mu{}_\nu \equiv \lambda^\mu{}_\nu$

$$= \frac{i}{2}\lambda_{\alpha\beta}\left[-i\eta^{\mu\alpha}\delta^\beta{}_\nu + i\eta^{\mu\beta}\delta^\alpha{}_\nu\right]$$

$$\Rightarrow [J^{\alpha\beta}]^\mu{}_\nu = -i\left(\eta^{\mu\alpha}\delta^\beta{}_\nu - \eta^{\mu\beta}\delta^\beta{}_\nu\right)$$

$$= -2i\,\eta^{\mu[\alpha}\delta^{\beta]}{}_\nu$$ ([ ] notation: see notes)

J are a basis for Λ

λ are parameters for Λ

W1(B)

Notes: 1.4
Tong: 1, 2.1

Recap: unify QM w/ relativity
RQM: fails (unitarity, particle # etc)
QFT: solves theese issues

Regular QM: Classical theory + quantise

$$q, p \to \hat{q}, \hat{p}$$

$$[q_i, p_j] = i\hbar\delta_{ij}$$

$$E = \frac{p^2}{2m} \quad \to \quad i\frac{\partial}{\partial t}\psi = \frac{-\nabla^2}{2m}\psi$$

$$E^2 = p^2 + m^2 \quad \to \quad \partial^2\psi + m^2\psi = 0$$

$\psi(\vec{x}, t)$: Wavefunction of Relativistic particle, mass m

t: parameter,
$\vec{x}$: observable, eigenvalue of $\hat{x}$ But ↑

QFT:

Classical field + quantise

$\phi(\vec{x}, t)$ $\quad$ x: label
x, t: both parameters
$\phi, \partial_\mu\phi$: dynamical variables are the fields

$\phi \to \hat{\phi}$: we'll see this soon

→ Turns out: - Solves above issues
- Describes arbitrary # of particles
→ creation + annhialation.

Elements of Classical Field Theory:

$$\delta S = 0$$

$$S = \int L \, dt$$

$$L = L(\phi, \partial_\mu \phi)$$

$\phi(x)$: assigns number to each x

scalar, vector, spinor etc.

Relativistic theory: S should be Lorentz Scalar

but then L cannot be, dt is not a scalar

$$\Delta t \to \gamma \Delta t$$
$$\Delta x \to \Delta x / \gamma$$

$$S = \frac{1}{c} \int \mathcal{L} \, \underbrace{d^4x}_{\text{scalar}} \quad , \quad L = \int \mathcal{L} \, d^3x$$

$\mathcal{L}$: Lagrangian density

↳ we just call it Lagrangian

$\mathcal{L}$ is scalar, d^4x puts $\vec{x}$, t on same footing

$$\boxed{d^4x = c\,dt\,dx\,dy\,dz}$$

$$\delta S = 0$$

ϕ and $\partial_\mu \phi$ are the dynamic variables!

$$= \int \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right] d^4x$$

NB: write like ϕ is a scalar: but valid in general $\phi \to \phi_a$

$$\to \sum_a \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \dots \quad \text{etc}$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right] + \cancel{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right]_{\text{Boundary}}} \to 0$$

$= 0$, for arbitrary $\delta\phi$

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}$$

↳ one for each field / component. $\phi \to \phi_a$

Important : $\mathcal{L} \to \mathcal{L} + \underbrace{\partial_\mu K^\mu}_{\text{total divergence}}$

goes to boundary term in S [Gauss' theorem]

$\phi \to 0$ at boundary (typical, but not 100% the case e.g. QCD...)

No impact on E.o.M

Define: densities!

$$\boxed{\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}} \qquad \left[p = \frac{\partial L}{\partial \dot{q}} \right]$$

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L}$$

$$\boxed{\mathcal{H} = \pi \dot{\phi} - \mathcal{L}} \qquad [H = p\dot{q} - L]$$

→ More meaningful when we consider symmetries + conservation laws in few few weeks.

$\phi(\vec{x}, t) \to \phi(t)$? Regular E-L equations!

What $\mathcal{L}$ are allowed?

- $\mathcal{L}$: ~~...~~ At most, ~~...~~ first-order time derivatives $\Rightarrow$ E.o.M have $\ddot{\phi}$ not $\dddot{\phi}$
  2) classical limit

- Local function of fields + deriv's

$$S = \int d^4x \, \mathcal{L}(\phi, \dot{\phi}) , \quad \text{not} \quad \underbrace{\int d^4x_1 d^4x_2 \, \mathcal{L}(\dots)}_{\text{Non-local}}$$

- real (ϕ may be complex!)

- Scalar. (with (1)↑ also $\Rightarrow$ 1st spatial derivatives)

+

Also:
Super-position
$\Rightarrow$ linear e.o.m
$\Rightarrow \mathcal{L} = \mathcal{L}(\phi^2)$ at most
(some, not all!)

How do fields transform

Scalar ϕ: maps a scalar to each x^μ

$$\phi(x^\mu) \longrightarrow \phi'(x'^\mu) = \phi'(\Lambda x)$$
$$= \phi(x)$$

or $\boxed{\phi'(x) = \phi(\Lambda^{-1}x)}$

Vector

$$A^\mu(x^\nu) \longmapsto A'^\mu(x'^\nu) = \Lambda^\mu{}_\nu A^\nu(x)$$

$$\boxed{A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)}$$

and so on

Example: Maxwell

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = (-\vec{E}, \vec{B})$$

$$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & -B_z & B_y \\ & & 0 & -B_x \\ & & & 0 \end{pmatrix}$$

$$S = \int (-m\,ds - q A_\mu dx^\mu)$$

$$\Rightarrow \boxed{\frac{dp^\mu}{ds} = q F^{\mu\nu} u_\nu} \quad \text{Lorentz force}$$

$c = 1$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

$$\Rightarrow \nabla \times \vec{E} = -\frac{\partial B}{\partial t}, \quad \nabla \cdot \vec{B} = 0$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \qquad \tilde{F}^{\mu\nu} = (-\vec{B}, -\vec{E})$$

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0}$$

F^2 - common Notation

$$\boxed{\mathcal{L} = \frac{-1}{4} \overbrace{F^{\mu\nu} F_{\mu\nu}}^{F^2} - J^\mu A_\mu}$$

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} = J^\nu} \Rightarrow \begin{pmatrix} \nabla \cdot \vec{E} = \underset{\sim}{J^0} \overset{\rho}{} \\ \nabla \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t} \end{pmatrix}$$

If unfamiliar:

1) Show F^2 is scalar

2) Derive EOM $\rightarrow$ proove $\dfrac{\partial F^2}{\partial(\partial_\mu A_\nu)} = 4 F^{\mu\nu}$

3) Prove $F^{\mu\nu}$ gauge invariant

$$A_\mu \rightarrow A_\mu + \partial_\mu \Theta(x^\nu)$$

How to quantise a theory

Generic Method

(1) Classical E.O.M $\Rightarrow$ operator equation

(2) Find most general solution to E.O.M

$\hookrightarrow$ promote integration constants to constant quantum operators

$\Rightarrow$ Gives quantum time evolution

(3) Impose canonical quantisation condition

$\Rightarrow$ Find commutation rules for operators in (2)

e.g. $[a,b] = i\hbar \{a,b\}_{PB}$ if classical analogue.

(5) Use operators $\nearrow$ to form full Hilbert Space.

Simplest Case : 1D Harmonic Oscillator

Classial

$$H = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} k x^2$$

$$L = \tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2$$

let $\omega = \sqrt{k/m}$, $q = \sqrt{k/\omega}\, x$

$$L = \frac{\dot{q}^2}{2\omega} - \frac{\omega}{2} q^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

$$\frac{1}{\omega}\ddot{q} = -q\omega \Rightarrow \boxed{\ddot{q} = -\omega^2 q}$$

$$q(t) = A e^{-i\omega t} + \underbrace{A^*}_{\text{Keep it Real}} e^{i\omega t}$$

$$p(t) = \frac{\partial L}{\partial \dot{q}} = \frac{1}{\omega}\dot{q} = -i\left(A e^{-i\omega t} - A^* e^{i\omega t}\right)$$

<u>Quantise</u> : $q, p \longmapsto \hat{q}, \hat{p}$

("step (2)")

Nb: I will stop putting hats ^ in places

$$\hat{p} = -i\hbar \frac{\partial}{\partial q}$$

$$[\hat{q}_i, \hat{p}_j] = i\hbar\, \delta_{ij}$$

1, only 1D problem

$$= [\hat{q}(t), \hat{p}(t)]$$

Heisenberge Picture operators

$$\hat{O}(t) = e^{\frac{i}{\hbar}Ht}\, \hat{O}\, e^{-\frac{i}{\hbar}Ht}$$

$$\dot{\hat{q}}(t) = \frac{i}{\hbar}[H, \hat{q}(t)] = \omega \hat{p}(t)$$

$$\dot{\hat{p}}(t) = \frac{i}{\hbar}[H, \hat{p}(t)] = -\omega^2 \hat{q}(t) = -\omega^2 \hat{q}(t)$$

$$\boxed{A \to \hat{A}, \quad A^* \to \hat{A}^\dagger}$$

(had $q(t) = Ae^{-i\omega t} + A^* e^{i\omega t}$)

For Reasons that will be clear,

$$A = \sqrt{\frac{\hbar}{2}}\, a$$

constant operators

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2}}\left(\hat{a}\, e^{-i\omega t} + \hat{a}^\dagger e^{+i\omega t}\right)$$

$$\hat{p}(t) = -i\sqrt{\frac{\hbar}{2}}\left(\hat{a}\, e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}\right)$$

$$\left(p = \frac{\partial \mathcal{L}}{\partial \dot{q}}\right)$$

$$[\hat{q}, \hat{p}] = (-i)\frac{\hbar}{2}\left(-[a, a^\dagger] + [a^\dagger, a]\right) = i\hbar$$

$$\Rightarrow \quad [a, a^\dagger] = 1$$

$$\hat{a} = \sqrt{\frac{1}{2\hbar}}\left(\hat{p}(0) - i\hat{q}(0)\right)$$

$$H = \hbar\omega\left(a^\dagger a + 1/2\right)$$

What do we do w/ New operators? Commutate them!

$$H = \hbar\omega(\underbrace{a^+ a}_{\hat{n}} + \tfrac{1}{2})$$

$$[a, a^+] = 1$$

$$[\hat{n}, \hat{a}^+] = a^+[a, a^+] = a^+$$

$$[\hat{n}, \hat{a}] = [a^+, a]a = -a$$

$$[H, a^+] = \hbar\omega[\hat{n}, a^+] = \hbar\omega a^+$$

$$[H, a] = -\hbar\omega a$$

$$[H, \hat{n}] = 0 \quad \Rightarrow \quad \boxed{|n\rangle : \hat{n} \text{ and } H \text{ eigenstates}}$$

$$\Rightarrow \quad \hat{n}|n\rangle = n|n\rangle$$ Defines 'n'

$$\Rightarrow \quad H|n\rangle = \varepsilon_n |n\rangle$$

$$\varepsilon_n = \hbar\omega(n + \tfrac{1}{2})$$

[$\hat{n} = a^+ a$ ⇒ Excitation # operator]

$a^+|n\rangle$?

$$\hat{n}(a^+|n\rangle) = (a^+\hat{n} + a^+)|n\rangle$$
$$= (n+1)(a^+|n\rangle)$$

$$\Rightarrow \quad a^+|n\rangle = c_n |n+1\rangle$$

Similarly $\quad a|n\rangle = \tilde{c}_n |n-1\rangle$

Raising / Lowering Operators.

By asserting Normalised in same way

$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

[Well-behaved theory must be bound from below!]

Define $|0\rangle \quad a|0\rangle = 0$

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle$$

Simple ODE

Quantum field theory?

Was the above a Q field theory?

Well... yes, but a very boring one.

$\Rightarrow$ Zero spatial degrees of freedom.

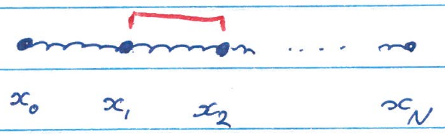
(interp. is different also:

QM: $\psi_n(x) = \langle x | n \rangle$

QFT: $\hat{\psi}(x, t) = \sum_{k\omega}$

1D chain of coupled Harmonic Oscillators:

Δx: 'lattice' spacing.



N spatial degrees of freedom

- q_i deviation of i^{th} osc. from equillib.
- Each has own potential: $= -\frac{1}{2}\eta q^2$
- Plus each pair: $-\frac{1}{2}\lambda(q_i - q_{i+1})^2$

$$\mathcal{L} = \sum_{i=0}^{N-1} \left[\frac{m}{2}\dot{q}_i^2 - \frac{\eta}{2} q_i^2 - \underbrace{\frac{\lambda}{2}}_{\text{Zero if uncoupled}} (q_i - q_{i+1})^2 \right]$$

We could write $q_i(t) = q(x_i, t)$

parameters

Also parameter!

x_i is merely a label, not a dyn. variable

q is the dynamical variable.

(Field theory: Hard, because spatial d.o.f are coupled)

Can define $p_n = \dfrac{\partial L}{\partial \dot{q}_n} = m\dot{q}_n$

$$H = \sum_n \left[\frac{p_n^2}{2m} + \frac{\eta}{2} q_n^2 + \frac{\lambda}{2} \underbrace{\Delta q_n^2}_{\Delta q_n = q_n - q_{n+1}} \right]$$

E-L $\qquad m\ddot{q}_n = -\eta q_n - \lambda(2q_n - q_{n+1} - q_{n-1})$

Solve? Make ansatz:

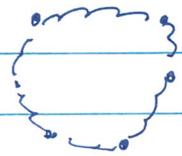
Now, also Lattice dependance

$$q_n \sim e^{-i\omega t} e^{ikn}$$

Sub in

$$-m\omega^2 = -\eta - \lambda \underbrace{(2 - e^{ik} - e^{-ik})}_{4\sin^2(\frac{k}{2})}$$

For simplicity: Periodic boundary



$$q_{n+N} = q_n$$

$$\Rightarrow e^{ikn} = e^{ik(n+N)} \Rightarrow e^{ikN} = 1$$

$$k_j = 2\pi \frac{j}{N} \qquad j = 0, 1, \dots, N-1$$

$\sim$ momentum-like

# of possible k values, Not 1 per lattice!

$$\omega_j = \sqrt{\frac{4\lambda}{m}\sin^2\left(\frac{k_j}{2}\right) + \frac{\eta}{m}}$$

Modes of excitation

$$\boxed{q_n(t) = \sum_{j=0}^{N-1} \left(A_j e^{ik_j n - i\omega t} + c.c. \right)}$$

General Solution

K: "Lattice" momentum

$$P_n = \frac{\partial L}{\partial \dot{q}_n} = \sum_j \left(-i\omega_j A_j e^{ik_j n - i\omega t} + cc \right)$$

<u>Quantise</u>

$$A \rightarrow \hat{A}$$

Impose Q condition: $[q_n, P_m] = i\hbar\,\delta_{nm}$ $\qquad [A, A^+] = ?$

Use $\displaystyle\sum_{n'=0}^{N-1} \exp\left(i\, \frac{2\pi\,[n-m]\,n'}{N} \right) = N\,\delta_{nm}$, $\quad k_j = \frac{2\pi j}{N}$

$$\Rightarrow \quad \delta_{nm} = \frac{1}{N} \sum_{n'} e^{i k_{n'} (n-m)}$$

After long algebra: $\boxed{[A_n, A_m^+] = \frac{\hbar}{2N\omega_n}\,\delta_{nm}}$

let $\quad a_n = \sqrt{\frac{2N\omega_n}{\hbar}}\, A_n$

$$\boxed{[a_n, a_m^+] = \delta_{nm}}$$

$$\boxed{H = \sum_{n=0}^{N-1} \hbar\omega_n \left(a_n^+ a_n + \tfrac{1}{2} \right) \qquad \omega_n = \omega(k_n)}$$

$$\omega_n = \sqrt{\frac{4\lambda}{m} \sin^2\left(\frac{k_n}{2}\right) + \frac{k}{m}}$$

Construct Hilbert Space:

N $\quad a^+$ operators

$$a_n|0\rangle = 0 \quad \text{for each} \quad n = 0 \ldots N-1$$

for each N degree of freedom, can excite

Construct Hilbert space:

$$|\underbrace{n_0}_{k_0}, \underbrace{n_1}_{k_1}, \ldots, n_{N-1}\rangle = (a_0^+)^{n_0} (a_1^+)^{\overbrace{n_1}^{\#\text{ of excitations of given } k}} \cdots (a_{N-1}^+)^{n_{N-1}} |0\rangle$$

(excluding Normalisation)

which k_j is excited

# of particles? not $\underline{N}$

$$\sum_{i=0}^{N-1} n_i \qquad n\text{'s from} \nearrow$$

$$\hat{n} = \sum_i a_i^+ a_i$$

Now: $\quad N \to \infty$

take continuum limit

"Lattice" spacing $\overset{\Delta x}{} \rightarrow 0$, $N \rightarrow \infty$

$$L = \underbrace{\Delta x \sum_n^{\infty}}_{\int dx} \left[\frac{m}{2\Delta x} \dot{q}_n^2 - \frac{\kappa}{2\Delta x} q_n^2 - \frac{\lambda \Delta x}{2} \underbrace{\frac{(\Delta q_n)^2}{\Delta x^2}}_{\left(\frac{\partial q}{\partial x}\right)^2} \right]$$

$$\frac{L}{\Delta x} \rightarrow \mathcal{L}$$

Really: $\left(L = \int dx \mathcal{L}\right)$

$\frac{m}{\Delta x} \rightarrow \mu$ (mass density)

$\lambda \Delta x \rightarrow \tau$ (tension / Young's modulus)

$\frac{\kappa}{\Delta x} \rightarrow m^2$ (stiffness / effective mass)

$q_n(t) \rightarrow q(x_n, t) \longrightarrow \phi(\vec{x}, t)$

$p \rightarrow \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \pi(\vec{x}, t)$

More Next Week

$[q_i, p_j] = i\delta_{ij}$

$\rightarrow [\phi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$

$$\mathcal{L} = \frac{\mu}{2}(\partial_t \phi)^2 - \frac{\tau}{2}(\partial_x \phi)^2 - \frac{m^2 \phi^2}{2}$$

3D: $\partial_x \rightarrow \nabla$,

$m = 0$, wave Equation, $c = \sqrt{\frac{\tau}{\mu}}$

For Lorentz scalar: $\mu = \tau$

Klein-Gordon

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2}$$

$(\mu = \tau = 1)$

E.O.M: $\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)$

$$\Rightarrow \boxed{(\partial^2 + m^2)\phi = 0}$$

→ Find $\mathcal{H}$

→ Find E.O.M

Next: quantise K.G

B.M. Roberts

Notes: 3.0 – 3.2
Tong: 2

Recap: Last time on QFT...

Fields: $q(t), p(t) \to \phi(t,\vec{x}), \pi(t,\vec{x})$

$$L = \int d^3x \, \mathcal{L}(\phi, \partial_\mu \phi \ldots)$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \qquad \left(\text{just like } p = \frac{\partial L}{\partial \dot{q}} \right)$$

$$\mathcal{H} = \sum_i \pi_i \dot{\phi}_i - \mathcal{L} \qquad \left(\text{like } H = \sum_i p_i \dot{q}_i - L \right)$$

Quantisation:

(Regular QM)

$$q, p \mapsto \hat{q}, \hat{p}, \qquad \hat{p} = -i\hbar \partial_q$$

$$\{q_i, p_j\}_{PB} = \delta_{ij} \longmapsto [\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}$$

$\hbar = 1$ usually from here

- We quantised single Harmonic oscilator $L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \Rightarrow H = \hbar\omega(a^\dagger a + \frac{1}{2})$

raising/lowering operators

- Then chain of N coupled H.O s
  $\hookrightarrow$ N spatial degrees of freedom

$$q_n(t) = \sum_j \left(A_j e^{i(k_j n - \omega t)} + cc \right)$$

n^{th} spatial location

j^{th} indep k solution

* Most General Solution

$$A \to \hat{A}$$

* promote to operator

$$[q_i, p_j] = i\hbar \delta_{ij}$$

* Impose Can. Quat.

$$\Rightarrow [a_i, a_j^\dagger] = \delta_{ij} \qquad \left(a = \sqrt{\frac{2N\omega}{\hbar}} A \right)$$

$$H = \sum_n \hbar\omega_n (a_n^\dagger a_n + \tfrac{1}{2})$$

Recap of last lecture:

1D Chain of N H-O

$$L = \sum_{n=0}^{N-1} \left[\underbrace{\frac{m}{2}\dot{q}_n^2}_{\text{Kinetic}} - \underbrace{\frac{\eta}{2} q_n^2}_{\substack{\text{Regular potential} \\ \text{"self coupling"}}} - \underbrace{\frac{\lambda}{2}(q_n - q_{n+1})^2}_{\text{Coupling to each neighbor}} \right]$$

E-L equations : $\frac{d}{dt}\left(\frac{dL}{d\dot{q}}\right) = \frac{dL}{dq}$ w/ Ansatz: $q_n \sim e^{-i\omega t} \underbrace{e^{ikn}}_{\text{Spatial dependence}}$

$$\Rightarrow -m\omega^2 = -\eta - \lambda(2 - e^{ik} - e^{-ik})$$

$$= -\eta - 4\lambda \sin^2\left(\frac{k}{2}\right) \quad \circledast \qquad \left(e^{ix} = \cos(x) + i\sin(x)\right)$$

For simplicity: periodic boundary conditions : $q_{n+N} = q_n \Rightarrow k_j = \frac{2\pi j}{N}$

allowed "modes" of oscillation

$$\circledast \Rightarrow \quad \omega_k = \sqrt{\frac{4\lambda}{m}\sin^2\left(\frac{k}{2}\right) + \frac{\eta}{m}}$$

$$\boxed{\begin{aligned} q_n(t) &= \sum_{j=0}^{N-1}\left(A_j e^{-i\omega_j t} e^{ik_j n} + cc \right) \\ p_n(t) &= \sum_{j=0}^{N-1} (-i\omega_j)\left[A_j e^{-i\omega_j t} e^{ik_j n} - cc \right] \end{aligned}}$$

n^{th} spatial location

j^{th} oscillation mode

Most general Solution

$p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$

- Impose quantisation: $[q_n, p_m] = i\hbar\,\delta_{nm}$
- Use Identity : $\sum_{n'=0}^{N-1} \exp\left(2\pi i \frac{n'(n-m)}{N}\right) = N\,\delta_{nm}$ with $k_j = \frac{2\pi j}{N}$
- Find: $[a_n, a_m^\dagger] = \delta_{nm}$ with $a_n = \sqrt{\frac{2N\omega_n}{\hbar}}\, A_n$

$$\Rightarrow \quad \boxed{H = \sum_{n=0}^{N-1}\left(a_n^\dagger a_n + \frac{1}{2}\right)\hbar\omega_n}$$

* Same as HO

$\hookrightarrow$ N summed

Construct Hilbert Space:

N a^+ operators

$a_n|0\rangle = 0$ for each $n = 0 \dots N-1$

for each N degree of freedom, can excite

Construct Hilbert space:

# of excitations of given K

$$|\underbrace{n_0}_{k_0}, \underbrace{n_1}_{k_1}, \dots, n_{N-1}\rangle = (a_0^+)^{n_0} (a_1^+)^{n_1} \dots (a_{N-1}^+)^{n_{N-1}} |0\rangle$$

(excluding Normalisation)

which k_j is excited

# of particles? not $\underline{N}$

$\sum_{i=0}^{N-1} n_i$ n's from ↗

$$\hat{n} = \sum_i a_i^+ a_i$$

Now: $N \to \infty$

take continuum limit

Continuum Limit:

"Lattice" spacing Δx $\to 0$, $N \to \infty$

$$L = \underbrace{\Delta x \sum_n^{\infty}}_{\int dx} \left[\frac{m}{2\Delta x_i} \dot{q}_n^2 - \frac{k}{2\Delta x_i} q_i^2 - \frac{\lambda \Delta x_i}{2} \underbrace{\frac{(\Delta q_i)^2}{\Delta x_i^2}}_{\left(\frac{\partial q}{\partial x}\right)^2} \right]$$

$$\frac{L}{\Delta x} \to \mathcal{L}$$

Really: $\left(L = \int dx\, \mathcal{L}\right)$

$\frac{m}{\Delta x} \to \mu$ (mass density)

$\lambda \Delta x \to \tau$ (tension / Young's modulus)

$\frac{k}{\Delta x} \to m^2$ (stiffness / effective mass)

$q_n(t) \to q(x_n, t) \longrightarrow \phi(\vec{x}, t)$

$p \to \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \pi(\vec{x}, t)$

More Next Week

$$\boxed{[q_i, p_j] = i\delta_{ij} \\ \to [\phi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x}-\vec{y})}$$

$$\mathcal{L} = \frac{\mu}{2}(\partial_t \phi)^2 - \frac{\tau}{2}(\partial_x \phi)^2 - \frac{m^2 \phi^2}{2}$$

3D: $\partial_x \to \nabla$, $\boxed{m = 0, \text{ wave Equation, } c = \sqrt{\frac{\tau}{\mu}}}$

For Lorentz scalar: $\mu = \tau$

Klein-Gordon

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2 \phi^2}{2}}$$

$(\mu = \tau = 1)$

E.O.M: $\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)$

$$\Rightarrow \boxed{(\partial^2 + m^2)\phi = 0}$$

→ Find $\mathcal{H}$

→ Find E.O.M

Next: quantise K.G

Klien-Gordon

$q_n(t) \rightarrow q(\underbrace{x_n}_{\text{just a label!}}, t)$

Continuum limit: $\Delta x = x_n - x_{n+1} \rightarrow 0$

$N \rightarrow \infty$

$q_n(t) \rightarrow \phi(x,t)$

$$\Rightarrow \boxed{\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2}$$

- One example of scalar field — real field ϕ
- Klein-Gordon Lagrangian

- Find equations of motion
- Quantise it

1) Check: is $\mathcal{L}$ a scalar?

ϕ is scalar field (definition)

m is scalar parameter (definition)

$m^2\phi^2$? ✓ yep

$\partial_\mu \phi$: co-vector

$\partial^\mu \phi$: vector

$\partial_\mu \phi \partial^\mu \phi$: Scalar!

EOM

- $$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

- $$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$ → careful!

If not used to notation, do slowly a few times!

Always check: Indices balanced?

Aside: if notation is confusing:

$$\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi = \frac{1}{2}\left[(\partial_0\phi)^2 - (\nabla\phi)^2\right]$$

$$= \frac{1}{2}\left[(\partial_0\phi)^2 - (\partial_x\phi)^2 - (\partial_y\phi)^2 - (\partial_z\phi)^2\right]$$

$$\frac{\partial}{\partial(\partial_\mu\phi)}\left[\frac{1}{2}\,\underbrace{\partial_\nu\phi\,\partial^\nu\phi}_{\text{contracted}}\right] \qquad : \mu = 0, 1, 2 \ldots$$

or

$$\frac{1}{2}\partial_\nu\phi\partial^\nu\phi = \frac{1}{2}(\partial_\nu\phi)\cdot\eta^{\nu\lambda}\cdot(\partial_\lambda\phi)$$

$$+ \quad \boxed{\frac{\partial}{\partial_\mu\phi}\partial_\nu\phi = \delta^\mu_\nu}$$

$$\Rightarrow \quad \frac{\partial}{\partial(\partial_\mu\phi)}\left[\frac{1}{2}\partial_\nu\phi\,\partial^\nu\phi\right] = 2\times\frac{1}{2}\times\partial$$

$$\Rightarrow \qquad \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

$$\left[\partial^2 + m^2\right]\phi = 0$$

Klein Gordon Equation

nb: $E^2 + \quad = p^2 + m^2$ $\qquad \hat{p}^\mu = i\hbar\partial^\mu$

Normal QM: $E \rightarrow i\hbar\frac{\partial}{\partial t}$, $p \rightarrow -i\hbar\nabla$ gives exactly the same

$\rightsquigarrow$ Relativistic particle (free) of mass m

RQM: ϕ : wavefunction , x : eigenstate of $\hat{x}$

$\hookrightarrow$ suffers issues we discussed before

QFT: $\hat{\phi}$ ~~...~~ field operator.

Quantise Klien Gordon

$\sim \frac{1}{\sqrt{2}}\left(\text{Klien} + \text{Klein}\right)$ No one really knows....

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi$$

$$\mathcal{H} = \pi\dot\phi - \mathcal{L}$$ ← replace $\dot\phi \to \pi$

$$\left(\mathcal{L} = \frac{1}{2}\underbrace{\dot\phi^2}_{\pi^2} - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi\right)$$

$$\mathcal{H} = +\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2$$

K.G equation $\quad (\partial^2 + m^2)\phi = 0$

> Consider $m=0$
> $\partial^2\phi = 0$ Wave equation
> plane waves $\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = 0$
> Speed c $(c=1)$

Ansatz: $\quad \phi \sim e^{-ik\cdot x}$

$\left(k\cdot x = k_\mu x^\mu = \underbrace{k_0}_{\omega} t - \vec{k}\cdot\vec{x}\right)$ Notation

Sub in: $\quad (\partial_t^2 - \nabla^2 + m^2)\,e^{-ik\cdot x}$

$$= -\omega^2 + |\vec{k}|^2 + m^2 = 0$$ Dispersion relation

$\omega_{\vec{k}} \equiv \underbrace{+}_{\text{Positive!}}\sqrt{\vec{k}^2 + m^2}$ or $k_\mu k^\mu = m^2$

> k^μ has 3, not 4 independant components
> Constrained by $k^2 = m^2$
> "on shell"

$$u_{\vec{k}}(x) \equiv e^{-ik\cdot x} = e^{-i\omega_k t + \vec{k}\cdot\vec{x}}$$

Most general solution

$$\boxed{\phi(x) = \int \underbrace{\frac{d^3\vec{k}}{(2\pi)^3}}_{\text{choice (Fourier)}}\left[A_{\vec{k}}\, u_{\vec{k}}(x) + \underbrace{A^*_{\vec{k}}}_{\text{Real. For } \mathbb{C}\text{, have 'B'} \Rightarrow \text{different.}}\, u^*_{\vec{k}}(x)\right]}$$

Aside:

Regular QM

→ Recognise $\phi^+_k(\vec{x},t) = u_{\vec{k}} = e^{-i\omega_k t + i\vec{k}\vec{x}}$

$\phi^-_k(\vec{x},t) = u^*_k = e^{i\omega_k t - i\vec{k}\vec{x}}$

⇒ Basis functions, Eigen states

$$\phi = \int \frac{d^3k}{(2\pi)^2} \left(A\phi^+ + A^*\phi^-\right)$$ general wavefunction

Shrö $\hat{H}\phi = \hat{\varepsilon}\phi$

$= i\frac{\partial}{\partial t}\phi$

$i\frac{\partial}{\partial t}\phi^+ = i(-i)\omega_k\phi^+ = +\omega_k\phi^+$ +ve "Energy" Solutions

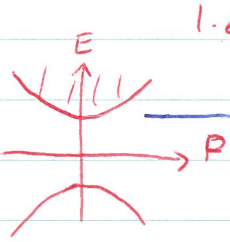
$i\frac{\partial}{\partial t}\phi^- = i(i)\omega_k\phi^- = -\omega_k\phi^-$ −ve "Energy" Solutions

* Old terminology

- We still use terminology sometimes

- But in QFT, this ↗ is not the energy
"−ve" E solutions will still have +ve energy.

Also: Plane waves are solutions only "on shell"
i.e. "on the mass shell", when $P_\mu P^\mu = E^2 + p^2 = m^2$



looks like a shell... I guess..

Now: Quantise Klein Gordon

$$q(t) \longrightarrow \hat{q}(t)$$

Regular QM

$$\{q_i, p_j\} = \delta_{ij} \longmapsto [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$$

Field? D. variables ϕ, π $\quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$

$$\phi \to \hat{\phi} = \hat{\phi}(t, \vec{x})$$

Quantum Postulate

$$\boxed{\begin{aligned} [\phi_a(\vec{x},t), \phi_b(\vec{y},t)] &= [\pi_a(t,\vec{x}), \pi_b(t,\vec{y})] = 0 \\ [\phi_a(t,\vec{x}), \pi_b(t,\vec{y})] &= i\hbar\, \delta^{(3)}(\vec{x}-\vec{y})\, \delta_{ab} \end{aligned}}$$

$$\phi \to \hat{\phi}, \quad A \to \hat{A}$$

Given these ↗, what is $[\hat{A}, \hat{A}^*]$? $\quad (\hbar = c = 1)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

By (one) convention: $A_k \to \frac{1}{\sqrt{2\omega_k}} a_k$ just normalisation

$$\boxed{\begin{aligned} \phi(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(\hat{a}_{\vec{k}}\, e^{-ik_\mu x^\mu} + \hat{a}^+_{\vec{k}}\, e^{+ik\cdot x} \right) \\ \pi(x) &= -i \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{\omega_k}}{\sqrt{2}} \left(\hat{a}_{\vec{k}}\, u_{\vec{k}}(x) - \hat{a}^+_{k}\, u^+_{\vec{k}}(x) \right) \end{aligned}}$$

Remember:
On-shell
$\Rightarrow \phi, k^\mu$ satisfies K.G
$\Rightarrow k_\mu k^\mu = m^2$

Drop hats.

Find $[a, a^+]$

Two methods: (a) "Brute force" compute $[\phi, \pi] = i\hbar\,\delta(\vec{x}-\vec{y})$

(b) Invert, $a = f(\phi, \pi)$, then directly have $[a, a^+]$

To proceed, make use of

$$\boxed{\int d^3x \; e^{i(\vec{p} \pm \vec{q})\cdot\vec{x}} = (2\pi)^3 \, \delta^{(3)}(\vec{p} \pm \vec{q})}$$

* If unfamiliar, just trust me for now
* We'll discuss this more soon

Take Fourier transform to invert, isolate a, a^+ :

NB: 4D Fourier factor/sign convention:

$$\tilde{f}(k) = \int d^4x \; e^{ikx} f(x) \qquad \begin{pmatrix} kx = k_\mu x^\mu \\ = Et - \vec{k}\cdot\vec{x} \end{pmatrix}$$

$$f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{f}(k)$$

$$\int d^3x \, \phi \, e^{-i\vec{p}\cdot\vec{x}} = \int \frac{d^3x \, d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \Big(a_k e^{-iE_kt} \underbrace{e^{i(\vec{k}-\vec{p})\cdot\vec{x}}}_{\delta^3(\vec{k}-\vec{p}) \,:\, \vec{k}\to\vec{p}} + a_k^+ e^{iE_kt} \underbrace{e^{-i(\vec{k}+\vec{p})}}_{\delta^3(\vec{k}+\vec{p}),\ \vec{k}\to-\vec{p}} \Big)$$

Nb: $E_{-\vec{p}} = E_{\vec{p}} = \sqrt{p^2+m^2}$

$$= \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} \, e^{-iE_pt} + a^+_{-\vec{p}} \, e^{iE_pt} \right)$$

Sorry: $\omega_k \equiv E_k$ Just Notation

Similarly

$$\int d^3x \; \pi(x) \, e^{-i\vec{p}\cdot\vec{x}} = (-i)\sqrt{\frac{E_p}{2}} \left(a_{\vec{p}} \, e^{-iE_pt} - a^+_{-\vec{p}} \, e^{iE_pt} \right)$$

↗ Valid for any t. Set $t=0$

Combine $\pm$ to isolate a_p, a^+_{-p}

$$\int \left[\sqrt{2\,\mathcal{E}_p}\;\phi(x) \;\pm i\sqrt{\frac{2}{\mathcal{E}_p}}\;\pi(x) \right] e^{\mp \vec{p}\cdot x}\, d^3x = \begin{cases} 2\, a_{\vec{p}} \\ 2\, a^{+}_{\vec{p}} \end{cases}$$

note: $t=0$ and $\mathcal{E}_{\vec{p}} = \mathcal{E}_{-p}$

$$a_{\vec{p}} = \frac{1}{\sqrt{2}} \int d^3x\; e^{-i\vec{p}\cdot\vec{x}} \left(\sqrt{\frac{\mathcal{E}_p}{1}}\;\phi(x) + \frac{1}{\sqrt{\mathcal{E}_p}}\, i\,\pi(x) \right)$$

$$a^{+}_{p} = \frac{1}{\sqrt{2}} \int d^3x\; e^{i\vec{p}\cdot\vec{x}} \left(\sqrt{\mathcal{E}_p}\;\phi(x) - \frac{1}{\sqrt{\mathcal{E}_p}}\, i\,\pi(x) \right)$$

$$[a_p, a^{+}_{q}] = \frac{1}{2} \int d^3x\, d^3y\; e^{-i\vec{p}\cdot\vec{x}}\, e^{i\vec{q}\cdot\vec{y}} \underbrace{[\quad \ldots \quad]}$$

only cross-terms surive

$$\sqrt{\frac{\mathcal{E}_p}{\mathcal{E}_q}}\,(-i)\, \underbrace{[\phi(x), \pi(y)]}_{i\hbar\, \delta^{(3)}(\vec{x}-\vec{y})} + i\sqrt{\frac{\mathcal{E}_q}{\mathcal{E}_p}}\, \underbrace{[\pi(x), \phi(y)]}_{-i\hbar\, \delta^{(3)}(\vec{x}-\vec{y})}$$

$$[a_{\vec{p}}, a^{+}_{\vec{q}}] =$$

$$= \frac{1}{2} \underbrace{\int d^3x\; e^{-i(\vec{p}-\vec{q})\vec{x}}}_{(2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q})} \Bigg(\underbrace{\sqrt{\frac{\mathcal{E}_p}{\mathcal{E}_q}} + \sqrt{\frac{\mathcal{E}_q}{\mathcal{E}_p}}}_{\Rightarrow 2} \Bigg)$$

$$\boxed{[a_{\vec{p}}, a^{+}_{\vec{q}}] = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q})}$$

(cf $[a_i, a^{+}_j] = \delta_{ij}$ for single H.O)

(Sim: $[a_p, a_q] = [a^{+}_p, a^{+}_q] = 0$ $(1-1)$ istead of $(1+1)$)

Step back: recap

- Partly motivated by ∞-chain of H.O

$\Rightarrow$ Klien Gordon Equation

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right)$$

$$\Rightarrow \quad (\partial^2 + m^2)\phi = 0$$

Quantised $\qquad\qquad \left(u = e^{-i k_\mu x^\mu} \right)$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2 E_{\vec{k}}}} \left(a_{\vec{k}}\, u_{\vec{k}}(x) + a^+_{\vec{k}}\, u^+_{\vec{k}}(x) \right)$$

$$\pi(x) = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{E_k}{2}} \left(a_k\, u_{\vec{k}}(x) - a^+_{\vec{k}}\, u^+_{\vec{k}}(x) \right)$$

$$[\phi(\vec{x}), \pi(\vec{y})] = i\hbar\, \delta^{(3)}(\vec{x} - \vec{y})$$ Imposed Canonical Quantisation

$$\Rightarrow \quad [a_{\vec{p}}, a^+_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \quad \Big| \quad \text{all others} = 0$$

What does it mean?

- Construct Hilbert space
  - $\hookrightarrow$ Same way as before for 1D H.O
- Need Hamiltonian

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} \qquad \left(\text{Remember: } \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \right)$$

$$= \frac{1}{2}\left[\pi^2 + (\nabla \phi)^2 + m^2 \phi^2\right]$$

- Write in terms of a, a^+

B.M. Roberts

PHYS4040: QFT – Lecture 4 – W2(B) – 2024

Notes: 3.2–3.3
Tong: 2
P&S: 2.3

Last time:

- Partly motivated by ∞ chain of H.O's
  $\Rightarrow$ Klein-Gordon Equation

$$\boxed{\mathcal{L} = \tfrac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right)}$$

$$\Rightarrow \quad (\partial^2 + m^2)\,\phi = 0$$

Regular QM: ϕ: wavefunction for relativistic scalar particle, mass m: $E^2 = p^2 + m^2$

Quantised:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \left[a_{\vec{k}}\, u_{\vec{k}}(x) + a^\dagger_{\vec{k}}\, u^*_{\vec{k}}(x) \right]$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi \quad \Rightarrow \quad \pi(x) = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{E_{\vec{k}}}{2}} \left[a_k u_k(x) - a^\dagger_k u^*_k(x) \right]$$

$$\left(u_{\vec{k}} = e^{-ik_\mu x^\mu} \quad \text{w/ } k_0 \equiv E_{\vec{k}} \quad \text{"on shell"} \right)$$

Imposed Canonical Quantisation:

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\hbar\, \delta^{(3)}(\vec{x}-\vec{y})$$

$$\Rightarrow \quad [a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q}) \qquad (\text{all others} = 0)$$

* What does it mean?

Goal:
- Construct Hilbert space (same as oscillator)
- Associate $a^\dagger, a$ w/ creation/annihilation of particles!
- Require Hamiltonian

$$\mathcal{H} = \pi\dot\phi - \mathcal{L}$$
$$= \tfrac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right]$$

- Diagonalise in $a/a^\dagger$ basis (i.e., write in terms of $a, a^\dagger$)

Check: Heisenberg equations of motion:

$$\dot{\phi} = i[H,\phi] \quad \overbrace{(=\pi)}^{\text{From before!}} \qquad \left(H = \int d^3x\, \mathcal{H}\right)$$
$$\dot{\pi} = i[H,\pi]$$

$$\dot{\phi}_{(x)} = \frac{i}{2}\int d^3y \left([\pi^2(y), \phi_{(x)}] + \underbrace{[(\nabla\phi(y))^2, \phi(x)]}_{0} + \underbrace{[m^2\phi(y)^2, \phi(x)]}_{0} \right)$$

$$\left(\begin{array}{l} [AB, C] = A[B,C] + [A,C]B \\ [\pi(x), \phi(y)] = -i\,\delta^3(\vec{x}-\vec{y}) \quad \to \text{only non-zero} \end{array} \right)$$

$$\dot{\phi}(x) = \frac{i}{2}\int d^3y \left(-2\,\pi(y)\, i\,\delta^{(3)}(\vec{x}-\vec{y}) \right)$$

$$= \pi(x) \qquad \text{- Not surprising!}$$

$\dot{\pi}$? Two non-zero terms:

$$\frac{i}{2}\int d^3y \left([m^2(\phi(y))^2, \pi(x)] \right)$$

$$= \frac{i}{2}\int d^3y \; 2m^2(\phi(y))\; i\,\delta(\vec{y}-\vec{x})$$

$$= -m^2\phi(\vec{x})$$

Also: $$\frac{i}{2}\int d^3y \left[(\nabla_y \phi(y))^2, \pi(x) \right]$$

$$= \frac{i}{2}\int d^3y \left(\nabla\phi(y)[\nabla\phi(y), \pi(x)] + [\nabla\phi(y), \pi(x)]\nabla\phi(y) \right)$$

only 'sees' $\vec{y}$

$$= \frac{i}{2}\int d^3y \left(\nabla\phi(y)\, \nabla_y\left([\phi(y), \pi(x)]\right) + \nabla\left([\phi,\pi]\right)(\nabla\phi) \right)$$

$$= \frac{i}{2}\int d^3y \left(-2(\nabla^2\phi(y))(i)\,\delta^{(3)}(\vec{y}-\vec{x}) \right) \qquad \text{use I.B.P}$$

$$= \nabla^2\phi(x)$$

$$\Rightarrow \quad \dot{\pi} = \nabla^2\phi - m^2\phi$$

$$\boxed{\begin{array}{l} \dot{\phi} = \pi \\ \ddot{\phi} = \dot{\pi} = (\nabla^2 - m^2)\phi \\ \Rightarrow (\partial_t^2 - \nabla^2 + m^2)\phi = 0 \Rightarrow \text{K-G Equation} \end{array}}$$

$$\mathcal{H} = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2 \right)$$

Notation:

$$\left[\begin{aligned} kx &= k_\mu x^\mu \\ &= k_0 t - \vec{k}\cdot\vec{x} \\ \text{w/ } k_0 &\equiv \varepsilon_k \end{aligned} \right]$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_k}} \left[a_k e^{-ikx} + a_k^+ e^{ikx} \right]$$

$$\pi(x) = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\varepsilon_k}{2}} \left[a_k e^{-ikx} - a_k^+ e^{ikx} \right]$$

* You re-do this for $\mathbb{C}$ field in A01!

Require:

[Set $t=0$

t-nonzero: Heisenberg!] $\quad e^{-ikx} \to e^{+i\vec{k}\cdot\vec{x}}$

$$\pi^2 = \pi^+\pi$$

$$= (i)(-i)\int \frac{d^3k\, d^3p}{(2\pi)^6} \frac{\sqrt{\varepsilon_k \varepsilon_p}}{2} \left(a_k^+ e^{-i\vec{k}\cdot\vec{x}} - a_k e^{i\vec{k}\cdot\vec{x}} \right)\left(a_p e^{i\vec{p}\cdot\vec{x}} - a_p^+ e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$= \int \frac{d^3k\, d^3p}{2(2\pi)^6} \sqrt{\varepsilon_k \varepsilon_p} \left[a_k^+ a_p e^{-i(\vec{k}-\vec{p})\cdot\vec{x}} - a_k^+ a_p^+ e^{-i(\vec{k}+\vec{p})\vec{x}} - a_k a_p e^{i(k+p)\cdot x} + a_k a_p^+ e^{+i(\vec{k}-\vec{p})x} \right]$$

Remember:

$$\boxed{\int d^3x\, e^{i(\vec{p}\pm\vec{q})\cdot\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{p}\pm\vec{q})}$$

$\delta(\vec{k}-\vec{p}) \Rightarrow p = k$

and $\delta(\vec{k}+\vec{p}) \Rightarrow p = -k$

Nb: $\varepsilon_{\vec{k}} = \varepsilon_{-\vec{k}} = +\sqrt{k^2 + m^2}$

$$H = \int d^3x\, \mathcal{H}$$

Kill p integral:

$$\boxed{\int d^3x\, \pi^2 = \int \frac{d^3k}{2(2\pi)^3} \varepsilon_{\vec{k}} \left(a_k^+ a_k - a_k^+ a_{-k}^+ - a_k a_{-k} + a_k a_k^+ \right)}$$

$\varepsilon_k \to \frac{\varepsilon_k^2}{\varepsilon_k}$ for convenience.

ϕ^2 term very similar:

$$\int d^3x\, m^2\phi^2 = \int \frac{d^3k}{2(2\pi)^3} \frac{m^2}{\varepsilon_{\vec{k}}} \left(a_k^+ a_k + a_k^+ a_{-k}^+ + a_k a_{-k} + a_k a_k^+ \right)$$

Finally: $\nabla\phi \Rightarrow \vec{k}$ pulled down

$$\nabla\phi(\overset{t=0}{0},\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\varepsilon_k}}\left[(i\vec{k})\,a_k\,e^{i\vec{k}\cdot\vec{x}} + (-i\vec{k})\,a_k^+\,e^{-i\vec{k}\cdot\vec{x}}\right]$$

(relative sign: same as π^2 w/ k^2
But: case $p \to -k$, extra $-$ve)

$$\int d^3x\,(\nabla\phi)^2 = \int \frac{d^3k}{2(2\pi)^3}\frac{|\vec{k}|^2}{\varepsilon_{\vec{k}}}\left[a_k^+a_k + a_k^+a_{-k}^+ + a_k a_{-k} + a_k a_k^+\right]$$

$$H = \int d^3x\,\frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right]$$

$$= \int \frac{d^3k}{4(2\pi)^3}\left[(a_k^+a_k + a_k a_k^+)\left(\frac{\overbrace{\varepsilon_k^2 + |\vec{k}|^2 + m^2}^{2\varepsilon_k^2}}{\varepsilon_k}\right) + (a_k^+a_{-k}^+ + a_k a_{-k})\,\underbrace{\left(\frac{-\varepsilon_k^2 + |\vec{k}|^2 + m^2}{\varepsilon_k}\right)}_{=0}\right]$$

$$|\vec{k}|^2 + m^2 \equiv \varepsilon_k^2 \;!$$

$$H = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,\varepsilon_{\vec{k}}\left[a_{\vec{k}}^+a_{\vec{k}} + \underbrace{a_{\vec{k}}a_{\vec{k}}^+}\right]$$

$$a_k a_k^+ = a_k^+ a_k + (2\pi)^3\delta$$

$$[a_k, a_{k'}^+] = 2\pi^3\delta(\vec{k}-\vec{k}')$$

(cf Harmonic Os)

$$a_k a_k^+ = [a_{k'}a_k^+] + a_k^+a_k$$

$$\boxed{H = \int\frac{d^3k}{(2\pi)^3}\,\varepsilon_{\vec{k}}\left(a_k^+a_k + \frac{1}{2}\underbrace{[a_k, a_k^+]}\right)}$$

Regular QM: $[a_k, a_k^+] = 1 \Rightarrow$ H.O

QFT: $[a_k, a_k^+] = (2\pi)^3\,\delta^{(3)}\underbrace{(\vec{k}-\vec{k})}_{0}$

Regular QM: Zero-point E: $\hbar\omega\frac{1}{2}$

QFT: $\hbar\omega\frac{\infty}{2}$ .....

$$\boxed{E_0 = \int\frac{d^3k}{(2\pi)^3}\frac{\varepsilon_{\vec{k}}}{2}[a_k, a_k^+]} \longmapsto \text{Our first infinity...}$$

* Ignoring ∞ for a moment:

QFT for Scalar $\mathcal{L} = \frac{1}{2}((\partial_\mu \phi)^\mu \phi + m^2\phi^2) \Rightarrow$ Klein-Gordon

↳ Reduces to continuous system of Harmonic oscillators labelled by $\vec{k}$

So, what about this infinity - maybe its not so bad? Can we skirt around the issue?

$$E_0 = \int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{E}_{\vec{k}}}{2} [a_k, a_k^+]$$

$$= \delta^{(3)}(0) \int d^3k \frac{\mathcal{E}_k}{2}$$

$$= \delta^{(3)}(0)\, 2\pi \int_0^\infty k^2 \mathcal{E}_k \, dk$$

$\mathcal{E}_k = \sqrt{k^2 + m^2}$

Note: We integrated over all space : ∞ Volume.

$$\delta^{(3)}(\vec{\Delta k}) = \frac{1}{(2\pi)^3} \underbrace{\int d^3x\, e^{i\vec{\Delta k}\cdot\vec{x}}}_{V} \quad w/ \quad \Delta\vec{k} \to 0$$

So, if we have $E_0 = V\tilde{\rho}_0$

maybe the Energy density, ρ, is finite?

Nope $\quad \frac{1}{2}\int_0^\infty \frac{\mathcal{E}_k}{2} \frac{d^3k}{(2\pi)^3} = 2\pi \int_0^\infty k^3 \sqrt{1 + (m/|k|)^2}\, dk$

Still ∞ !

Sum of ground-state energies for ∞ number of Harmonic Oscillators!

Resolution: - Only measure energy differences
- Simply discard

Aside: What about gravity, which should 'see' total E density?
Maybe, we solved Dark Energy problem?

$$\rho_{vac} = \frac{2\pi}{(2\pi)^3}\int_0^{\cancel{\infty}\to\Lambda} k^2\sqrt{k^2+m^2}\,dk \approx \frac{\Lambda^4}{16\pi^2}$$

Introduce cut-off Λ

$\Lambda \sim M_{Planck} =$ $\qquad M_{Planck} = \sqrt{\frac{\hbar c}{G}} \sim 10^{19}$ GeV

$$\rho_v \sim 10^{110}\ eV^4$$

Dark Energy? $\quad \rho_{DE} \approx 10^{-11}\ eV^4$

"famous" 120 orders-of-magnitude disagreement.

even if $\Lambda \sim m_e \sim 0.5$ MeV, $\rho_v \sim 10^{20}\ eV^4$

"Resolution" Simply discard constant terms

$$\boxed{H = \int \frac{d^3k}{(2\pi)^3}\, \varepsilon_{\vec{k}}\, a^+_{\vec{k}} a_{\vec{k}}}$$

See Tong 2.3

Systematic way:

Normal Ordering:

move all 'a$^+$' to left

$:\hat{O}:$

↑ signifies normal ordered

How to interpret physically? $\rightsquigarrow$ Hilbert Space - Fock space

See
Notes: 3.2.2
Tong: 2.4

- Same form as Harmonic oscillator
- Same $[a, a^\dagger]$

$\rightarrow$ $a_{\vec{p}}, a^\dagger_{\vec{p}}$ define algebra for Harmonic oscillators for all $\vec{p}$

$\rightarrow$ Proceed in exactly the same way

Postulate a vacuum $|0\rangle$:

$$\boxed{a_{\vec{p}}|0\rangle = 0}$$

Defines $|0\rangle$

- Hilbert space for KG field : spanned by States of the form

$$\boxed{|\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N\rangle = a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} \ldots a^\dagger_{\vec{p}_N}|0\rangle}$$

- For any (finite) combination of continuous $\vec{p}$
- $N < \infty$

$\rightarrow$ Each is called a Fock state

- Generalisation to varying # of "particles"

(recognise $|\vec{p}\rangle$ as 1 "particle" of momentum $\vec{p}$ $\rightarrow$ soon!)

Normalisation:

$\langle 0|0\rangle = 1$ by convention

w/ $|\vec{p}\rangle = a^\dagger_p|0\rangle$

$$\langle \vec{p}|\vec{q}\rangle = \langle 0|a_p a^\dagger_q|0\rangle$$

$$= (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$$

* Return to this!

$\rightarrow$ Same as plane-waves in regular QM

$\rightarrow$ Indeed, thats what they are!

Particles

Just as in H.O case :

$$[H, a^+_{\vec{p}}] = \mathcal{E}_{\vec{p}}\, a^+_{\vec{p}}$$
$$[H, a_{\vec{p}}] = -\mathcal{E}_{\vec{p}}\, a_{\vec{p}}$$

see W1(B) pg.8

$$[H, n_p] = 0 \quad , \quad n_p = a^+_p a$$

1 "particle" state $|\vec{p}\rangle = a^+_p |0\rangle$ is E eigen states

Energy : $\boxed{H|\vec{p}\rangle = \mathcal{E}_{\vec{p}}|p\rangle}$

$$[H, a^+_{\vec{p}}]|0\rangle = \mathcal{E}_{\vec{p}}\, a^+_p |0\rangle$$
$$H a^+_p |0\rangle - \underbrace{a^+_{\vec{p}} H |0\rangle}_{0} = \mathcal{E}_p |p\rangle$$
$$\Rightarrow H|p\rangle = \mathcal{E}_p |p\rangle$$

$\mathcal{E}_p = +\sqrt{p^2 + m^2}$ → Rel. energy of particle of mass m!
$(\mathcal{E}^2 = p^2 + m^2)$

⇒ Recognise $|p\rangle$ as momentum (+ Energy) eigen state of Relativistic particle of mass m

Momentum

Classical Fields : $\vec{P} = -\int d^3x\, \pi (\nabla\phi)$

See 'Fields' notes : we'll derive in few weeks

↙ Quantum version : just from $\hat{\pi}, \hat{\phi}$

$$\boxed{\hat{P} = \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a^+_p a_p}$$

$$\boxed{\hat{P}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle} \qquad \text{OK!}$$

Home work: show from →

Number operator

Simple H.O : $\hat{n} = a^\dagger a$

$\Rightarrow$ number operator

QFT

$$\boxed{N = \int \frac{d^3\vec{p}}{(2\pi)^3} \, a^\dagger_{\vec{p}} \, a_{\vec{p}}}$$

Spin:

- Later, we will define angular momentum operator

$$J^i = \varepsilon^{ijk} \int d^3x \, (\mathcal{J}^0)^{jk}$$

From Lec 1!

$$\langle \vec{p} | J | \vec{p} \rangle \big|_{p=0} = 0$$

$\Rightarrow$ No intrinsic $\vec{J}$

$S = 0$!

Statistics

Fock states:

$$|\vec{p}_1, \vec{p}_2, \ldots \vec{p}_N \rangle = a^\dagger_{p_1} \ldots a^\dagger_{p_N} |0\rangle$$

$\rightarrow$ Since $[a^\dagger_p, a^\dagger_q] = 0$, re-ordering changes nothing!

$\rightarrow$ Symmetric under particle exchange

$$|\vec{p}, \vec{q}\rangle = + |\vec{q}, \vec{p}\rangle$$

$\Rightarrow$ Bosons

Lorentz Invariant Normalisation

(see notes: 3.3
P&S : end of 2.3)
Tong: 2.4.1

$\langle 0 | 0 \rangle = 1$ great

Also: $\langle \vec{p} | \vec{q} \rangle \equiv \langle 0 | a_p a_q^\dagger | 0 \rangle = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$

Scalar? No!

$$\int \underbrace{d^3 p}_{\text{Not Scalar}} \, \delta^{(3)}(\vec{p}) = \underbrace{1}_{\text{Scalar}} \quad \Rightarrow \delta^{(3)}(p) \text{ is } \underline{\text{not}} \text{ Scalar}$$

Guess: dimensional analysis $[\delta^{(3)}] = \dfrac{1}{E^3}$ $\quad$ $\dfrac{E\, \delta^{(3)}(p)}{\cancel{E_0}}$ is scalar

know $d^4 p$ is scalar

$$\int \frac{d^3 p}{(2\pi)^3} \, g(\mathcal{E}_p, \vec{p}) \sim \int \frac{d^4 p}{(2\pi)^4} \, g(p^\mu) \, (2\pi) \, \delta(p_\mu p^\mu - m^2) \, \underbrace{\Theta(p_0)}_{\text{ensures } p_0 > 0}$$

Use ugly relation

$$\boxed{\delta(f(x)) = \sum_{x_0} \frac{\delta(x - x_0)}{|f'(x_0)|}}$$

all zero's, $f(x_0) = 0$

→ More on this next week.

$$\delta(p_0^2 - \vec{p}^{\,2} - m^2)\Big|_{p_0 > 0} = \frac{\delta(p_0 - \mathcal{E}_{\vec{p}})}{2\, \mathcal{E}_{\vec{p}}}$$

zeros: $\pm \mathcal{E}_{\vec{p}}$

Θ: chooses +ve

$$\rightarrow \underbrace{\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\mathcal{E}_{\vec{p}}}}_{\text{Lorentz invariant } \vec{p} \text{ integral}} \; \underbrace{g(\mathcal{E}_p, \vec{p})}_{\text{May or may not be Lorentz invariant.}}$$

So, define: $\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 \underbrace{2 E_{\vec{p}}\, \delta^{(3)}(\vec{p}-\vec{q})}_{\text{Lorentz Invar}}$

Re define: $\tilde{a}(\vec{p}) \longrightarrow \sqrt{2 E_{\vec{p}}}\, a_{\vec{p}}$

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a^{+}_{p} |0\rangle$$

$$= \tilde{a}^{+}(\vec{p})\, |0\rangle$$

Now:

$$[\phi(\vec{x}), \pi(\vec{y})] = i\, \delta^{(3)}(x-y) \qquad \text{same}$$

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \underbrace{\frac{1}{2E_{\vec{p}}}}_{\text{Before}\ :\ \frac{1}{\sqrt{2E}}} \left(\tilde{a}_p\, e^{i\vec{p}\cdot\vec{x}} + \tilde{a}^{+}_p\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\pi(\vec{x}) = \frac{-i}{2} \int \frac{d^3 p}{(2\pi)^3} \left(\tilde{a}_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - \tilde{a}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$[\tilde{a}_{\vec{p}}, \tilde{a}^{+}_{\vec{q}}] = 2E_{\vec{p}}\, (2\pi)^3\, \delta(\vec{p}-\vec{q})$$

$$\hat{H} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}\, E_{\vec{p}}\, \tilde{a}^{+}_{\vec{p}} \tilde{a}_{\vec{p}}$$

Be cautious.

Notes: $\tilde{a}$ written just as a

P & S: uses 'old' a

Tong: defines $\hat{a}_{\vec{p}} = a(\vec{p})$

Sorry...