4 Relativistic Quantum Mechanics for the Impatient

This set of notes should best be thought of as a *companion* to a good textbook, and is by no means complete. I can recommend (among many other great books):

- J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics* (2011). The final chapter of this book has a very nice, though breif, introduction to relativistic quantum mechanics.
- J. J. Sakurai, *Advanced Quantum Mechanics* (Cambridge University Press, 1967). An old book (written before electroweak unification), and uses some outdated notations. However, Sakurai's masterful treatment of the topic is still an excellent read.
- D. J. Griffiths, *Introduction to Elementary Particles* (Wiley, Dec. 1987). A solid text introducing particle physics and the basis of quantum field theory.
- V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics* (Pergamon, Oxford, 1982) *[Landau & Lifshitz, Volume 4]*. An old and technical book, with some very nice examples.
- M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Westview Press, 1995) (particularly chapters 3-4). A more advanced book aimed at developing the full quantum field theory, it has a very thorough coverage of the introduction to quantum electrodynamics.

4.1 Need for a relativistic theory

The non-relativistic Schrodinger equation for a free electron:

$$\left(\frac{\hat{\boldsymbol{p}}^2}{2m} - \varepsilon\right)\psi = 0 \qquad \text{or} \qquad \left(\frac{\hbar^2\nabla^2}{2m} + i\hbar\partial_t\right)\psi = 0, \tag{1}$$

is clearly not a Lorentz invariant equation – it's second-order in spatial derivatives, while only first-order in time. Historically, this may seem strange. Einstein's theory of relativity was well known and understood by the time Schrodinger wrote down his equation.

In fact, one may readily form a second-order wave equation that *is* Lorentz invariant[1], starting from the famous relativistic energy-momentum relation:

$$\mathcal{E}^2 = \boldsymbol{p}^2c^2 + m^2c^4, \tag{2}$$

(we use $\mathcal{E}$ here, as opposed to ε used in Eq. (1), as a reminder than this includes the rest energy).

Re-writing Eq. (2) in the relativistic notation [see Eq. (6)], and swapping from here on to use units where $\hbar = 1$ and $c = 1$:

$$p_\mu p^\mu - m^2 = 0. \tag{3}$$

(Note that we use the correct '-2' metric (6).) We can make a quantum wave equation by "quantising" the equation; promoting the four-momentums to operators:

$$\left(\hat{p}_\mu\hat{p}^\mu - m^2\right)\psi = 0, \qquad \text{or} \qquad \left(\partial_\mu\partial^\mu + m^2\right)\psi = 0. \tag{4}$$

This is the *Klein-Gordon* equation. (This is for a free particle; in the presence of an external field, a potential term, V, also appears as usual.) It turns out that this is the correct quantum equation to describe scalar

[1] I'm using the phrase 'Lorentz invariant' here rather sloppily, but the meaning should be clear

(that is, spinless) particles. In the non-relativistic limit ($\mathcal{E} \approx m$, $\varepsilon \ll m$), it recovers the usual Schrodinger equation. It turns out, however, that due to a chance cancellation between terms (relativistic correction to the kinetic energy, which the Klein-Gordon equation accounts for, and the spin-orbit effect, which it doesn't), the hydrogen energy levels predicted by the Klein-Gordon equation are actually *worse* than those predicted by the Schrodinger equation. Indeed, the story goes that Schrodinger himself found the relativistic Klein-Gordon equation before discovering his eponymous equation, but discarded it for this reason.

For a free particle (no potential term), the solutions to the KG equation are plane waves:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(\eta(\boldsymbol{p}) e^{i-px} + \chi(\boldsymbol{p}) e^{i-px} \right), \tag{5}$$

where η and χ are complex amplitudes. (We use ϕ here just to distinguish from the solution to the Dirac equation in the next section.) Even though this is not the wave equation for electrons, as we'll see, the KG solutions are useful in a number of contexts.

Notation:

$$\begin{aligned}
x^\mu &= (x_0,\, \boldsymbol{x}) & \hat{\boldsymbol{p}} &= -i\hbar\boldsymbol{\nabla} \\
x_\mu &= (x_0,\, -\boldsymbol{x}) & \hat{\varepsilon} &= i\hbar\partial_t \\
p^\mu &= (p_0,\, \boldsymbol{p}) = (\mathcal{E},\, \boldsymbol{p}) & \hat{p}_\mu &= i\hbar\partial_\mu \\
\partial_\mu &= \frac{\partial}{\partial x^\mu} = (\partial_t,\, \boldsymbol{\nabla}) & \partial_\mu\partial^\mu &\equiv \partial^2 \equiv \Box \quad \text{(d'Alembertian)} \\
\not{a} &= \gamma^\mu a_\mu & g_{\mu\nu} &= \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) \\
\mu &= 0,1,2,3 \quad \text{(Greek)} & x_\mu y^\mu &= \sum_{\mu=0}^{3} x_\mu y^\mu \quad \text{(Einstein summation)} \\
a &= 1,2,3 \quad \text{(Latin)}
\end{aligned} \tag{6}$$

4.2 Dirac equation

While the Klein–Gordon (KG) equation is a relativistic quantum theory, it is not the appropriate equation for particles with spin, such as electrons.

Dirac sought to find a quantum equation that obeyed the relativistic energy-momentum relation, but that was first-order in derivatives. The route he took was to factorise the KG equation:

$$p_\mu p^\mu - m^2 = (\Gamma^\mu p_\mu + m)\,(\gamma^\nu p_\nu - m) = 0 \tag{7}$$

where Γ and γ are arbitrary factorisation constants, which are to be found. Expanding, and matching terms, we get:

$$\begin{aligned}
p_\mu p^\mu - m^2 &= (\Gamma^\mu p_\mu + m)\,(\gamma^\nu p_\nu - m) \\
&= \Gamma^0\gamma^0 p_0 p_0 + \Gamma^1\gamma^1 p_1 p_1 + \ldots \\
&+ (\Gamma^0\gamma^1 + \Gamma^1\gamma^0) p_0 p_1 + (\Gamma^0\gamma^2 + \Gamma^2\gamma^0) p_0 p_2 + \ldots \\
&+ (-\Gamma^0 + \gamma^0) p_0 m + (-\Gamma^1 + \gamma^1) p_1 m + \ldots \\
&- m^2.
\end{aligned} \tag{8}$$

We equate this with the relativistic relation Eq. (2). Since there should be no cross terms of the form $\boldsymbol{p}\,m$, we must have $\Gamma = \gamma$ (from the fourth line). And since there should be no cross-terms of the form $p_\mu p_\nu$ for $\mu \neq \nu$, we must have $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$ (from the third line). Finally, from the second line, we see that we must have $(\gamma^0)^2 = 1$, and $(\gamma^a)^2 = -1$. We can combine these rules into a single equation

$$\{\gamma^\mu,\, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{9}$$

which defines the γ factors ($\{a,b\} = ab + ba$ is the anti-commutator). Clearly, there are no solutions to Eq. (9) that are real or complex numbers. It turns out, the simplest solutions are 4×4 matrices. The solutions are known as *gamma matrices*, or *Dirac matrices*.

Two common sets of solutions (representations) are the chiral (or Weyl) representation, and the Dirac representation:

$$\begin{array}{ll} \text{Chiral rep.:} & \text{Dirac rep.:} \\ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix} & \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \end{array} \tag{10}$$

where '1' is understood to be the 2×2 identity matrix, and σ^a are the 2×2 Pauli spin matrices ($[\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma^k$). Different representations are convenient in different applications; for example, the chiral basis is nice in relativistic problems involving free electrons, since in this representation, the chiral (or spin) projections of the wavefunction are simple. While the Dirac representation is nice for atomic physics, e.g., since the wavefunction has a simple form in the non-relativistic limit. We will mention now that it is convenient to also define the supplementary matricies:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \qquad \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{11}$$

with $(\gamma^5)^2 = 1$, $\{\gamma^5, \gamma^\mu\} = 0$.

With this, we may finally write down the famous Dirac equation:

$$\boxed{(i\gamma^\mu\partial_\mu - m)\,\psi = 0} \tag{12}$$

(or, $i\partial\!\!\!/\psi = m\psi$). Since the gamma matrix is four-by-four, the solution ψ is now a four-compnent object, called a Dirac spinor (or a Dirac bi-spinor):

$$\psi = \begin{pmatrix} \psi_1, & \psi_2, & \psi_3, & \psi_4 \end{pmatrix}^{\text{Tr}}. \tag{13}$$

Very roughly speaking, the four components correspond to the four degrees of freedom for the electron: spin up/down, and positive/negative energy solutions (electron/positron).

Notice that, by definition of the γ matrices, a 'second' operation of the (complex conjugate of the) Dirac operator results in the KG equation:

$$(i\gamma^\mu\partial_\mu + m)\,(i\gamma^\nu\partial_\nu - m)\,\psi = 0 \tag{14}$$

$$\implies \left(\partial^2 + m^2\right)\psi = 0. \tag{15}$$

Therefore, for any solution to the Dirac equation, each of the four Dirac components is automatically also a solution to the KG equation.

One can form a Hamiltonian equation readily from the Dirac equation. Of course, we must fix a reference frame to do this (the energy is not a Lorentz invariant). So, seperating the time and spatial derivatives (with $0 \to t$, $1 \to x$ etc.,), we have:

$$\left(i\gamma^0\partial_t + i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} - m\right)\psi = 0. \tag{16}$$

Multiply by γ^0, and re-arrange

$$\left(-i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + \gamma^0 m\right) = i\partial_t\psi. \tag{17}$$

This is in the form of the Schrodinger equation, with "Dirac" Hamiltonian:

$$H_D = \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p} + \gamma^0 m. \tag{18}$$

In Dirac's original notation (which is still commonly used), this was expressed: $H_D = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m$. We'll see how potential (interaction) terms are introduced later.

As an aside, we'll mention here that it is common to also define:

$$\gamma^5 \equiv i\gamma^0\gamma^2\gamma^2\gamma^3, \quad \text{and} \quad \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu], \tag{19}$$

with $(\gamma^5)^2 = 1$, $\{\gamma^5, \gamma^\mu\} = 0$.

4.3 Plane wave solutions: basis spinors

We'll consider first the simplest case, where there is no spatial dependence in the Dirac spinor, $\psi \neq \psi(x)$. In this case,

$$(i\gamma^0\partial_0 - m)\psi = 0. \tag{20}$$

We'll use the Weyl basis, and re-write Eq. (13) as

$$\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}. \tag{21}$$

where $\psi_L = (\psi_1, \psi_2)$ and ψ_R are two-component spinors (the meaning of 'L' and 'R' will be clear later). In the Weyl representation, this leads to two coupled equations:

$$i\partial_t\psi_L - m\psi_R = 0 \tag{22}$$
$$i\partial_t\psi_R - m\psi_L = 0. \tag{23}$$

It's simple to verify that there are four linearly independent solutions, which we denote $u_\pm$ and $v_\pm$:

$$u_+ \propto \begin{pmatrix}1\\0\\1\\0\end{pmatrix} e^{-imt}, \quad u_- \propto \begin{pmatrix}0\\1\\0\\1\end{pmatrix} e^{-imt}, \quad v_+ \propto \begin{pmatrix}1\\0\\-1\\0\end{pmatrix} e^{+imt}, \quad v_- \propto \begin{pmatrix}0\\1\\0\\-1\end{pmatrix} e^{+imt}. \tag{24}$$

The proportionality indicates we haven't chosen a normalisation yet.

A simple application of the Dirac Hamiltonian shows that

$$Hu^s = mu^s, \quad \text{and} \qquad Hv^s = -mv^s. \tag{25}$$

The us are then seen to be positive energy solutions (we'll call electrons), while the vs are seen to be negative energy solutions (we'll call them positrons). The s indices refer to the other degree of freedom, we'll call 'spin'. We'll name the $\pm$ solutions spin "up" and spin "down" solutions, respectively; the connection of these solutions to the idea of spin will be made clear in the next sections.

The presence of negative energy states immediately posses a deep issue: why don't electrons decay into the lower-energy positron states? Dirac postulated we could overcome this issue by assuming all the negative energy states were occupied, inventing what was termed the "Dirac sea". Positrons could then be interpreted as 'holes' in the Dirac sea.

As you'll see a little later, in a proper quantum field theory, it will turn out that positrons have the same (positive) mass as electrons, but instead have a negative charge, and the Dirac sea is no longer required (though is still a useful concept, particularly in low-energy systems).

We will now introduce the *adjoint* spinor, defined:

$$\bar\psi \equiv \psi^\dagger\gamma^0. \tag{26}$$

The reason for its introduction is that the quantity $\bar\psi\psi$ is a Lorentz invariant scalar, while $\psi^\dagger\psi$ is not. It turns out that $\bar\psi\gamma^\mu\psi$ scales as a Lorentz 4-vector, and so $\psi^\dagger\psi$ is actually just one component of a Lorentz vector, which is of course not invariant. This is not proved here, but just states. Now that we have a Lorentz scalar, we can make a sensible choice for the normalisation condition. While $\bar u u = 1$ would be a reasonable choice, it is common and convenient to instead define the normalisation for the above basis spinors as

$$\bar u u = 2m, \qquad \bar v v = -2m. \tag{27}$$

The solutions for the more general case of a free electron can readily be obtained in a number of ways (for example, by assuming a plane-wave ansantz, or by simply boosting our stationary solutions to a moving coordinate frame). The general solutions to the Dirac equation can be written as linear combintations of plane waves, which are of the form:

$$\psi(x) = u(p)e^{-ip_\mu x^\mu}, \quad \text{and} \quad \psi(x) = v(p)e^{+ip_\mu x^\mu} \tag{28}$$

for the positive and negative energy states, respectively, with

$$u^s(p), v^s(p) = \begin{pmatrix} \sqrt{\mathcal{E} - \boldsymbol{\sigma}\cdot\boldsymbol{p}}\,\chi^s \\ \pm\sqrt{\mathcal{E} - \boldsymbol{\sigma}\cdot\boldsymbol{p}}\,\chi^s \end{pmatrix} \tag{29}$$

where χ^s is a two-component spinor, eigenstates of, e.g., σ_z (e.g., $\chi^+ = [1, 0]$). The basis spinors have the orthonormality relations:

$$\begin{aligned} \bar{u}^r(p)u^s(p) &= 2m\delta_{rs} \\ \bar{v}^r(p)v^s(p) &= -2m\delta_{rs} \\ \bar{u}^r(p)v^s(p) &= \bar{v}^r(p)u^s(p) = 0. \end{aligned} \tag{30}$$

They also obey the (spin-space) completeness relations for the outer products:

$$\begin{aligned} \sum_{s=\pm} u^s(p)\bar{u}^s(p) &= \not{p} + m \\ \sum_{s=\pm} v^s(p)\bar{v}^s(p) &= \not{p} - m. \end{aligned} \tag{31}$$

These are not proved here.

We can also expand general solutions to the Dirac equation in terms of plane waves:

$$\begin{aligned} \psi(x) &= \sum_s \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[b_p^s u^s(p)e^{-ipx} + c_p^{*s}v^s(p)e^{ipx}\right], \\ \bar{\psi}(x) &= \sum_s \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[b_p^{*s}\bar{u}^s(p)e^{ipx} + c_p^{s}\bar{v}^s(p)e^{-ipx}\right], \end{aligned} \tag{32}$$

where b and c are complex amplitudes (which pairs we chose to be conjugated is simply convention).

The total energy in the field is:

$$\langle\psi|H_D|\psi\rangle = \int d^3x\bar{\psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\psi = \sum_s \int \frac{d^3p}{(2\pi)^3}E_{\boldsymbol{p}}(b_p^s b_p^{*s} - c_p^s c_p^{*s}). \tag{33}$$

Again, we see that the 'u' particles (electrons) contribute positively to the energy, while the 'v' particles (positrons) contribute negatively. We'll re-address this issue when we properly quantise the Dirac field.

4.4 Dirac Lagrangian and electrodynamics

The Lagrangian corresponding to the free-field Dirac equation is

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \tag{34}$$

You can see this with a simple application of the Euler-Lagrange equation:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \frac{\partial\mathcal{L}}{\partial\phi}. \tag{35}$$

Taking $\phi = \bar{\psi}$ results in the Dirac equation.

We will now consider the effect of a tranformation of the form:

$$\psi \to \psi' = \psi e^{-iq\theta}. \tag{36}$$

This corresponds to a U(1) symmetry. For a *global* transformation, θ is everywhere constant. It's immediately clear on substitution into (34) this leaves the Lagrangian unchanged, so the Dirac Lagrangian has a global U(1) symmetry.

But what about local invariance, with $\theta = \theta(x)$? In this case, the derivatives introduce new terms:

$$\partial_\mu \psi \to (\partial_\mu) e^{-iq\theta(x)} + \psi(-iq)(\partial_\mu \theta) e^{-iq\theta(x)}, \tag{37}$$

and so the Lagrangian *is* changed:

$$\mathcal{L}' = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi + q\bar{\psi}\gamma^\mu \psi \partial_\mu \theta. \tag{38}$$

As it stands, we do not have a local U(1) symmetry.

There is no real reason the Lagrangian *should* have this particular symmetry (besides from the analogy with classical physics, which we will see shortly). However, the study of symmetries such as these has been very fruitful in the invention of physics. Moreover, as we shall see, enforcing invariance under this symmetry will lead to the introduction of several important physical phenomena.

Suppose we introduce a second field, A_μ, which transforms as

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \theta, \tag{39}$$

so that the un-transformed Lagrangian is

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \left(\partial_\mu + iqA_\mu\right)\psi - m\bar{\psi}\psi. \tag{40}$$

Now, it is a quick exercise to see that this new term cancels out the extra θ derivative on the transformation, meaning we have $\mathcal{L} = \mathcal{L}'$, and the local U(1) symmetry is restored. It's common to introduce notation

$$D_\mu = \partial_\mu + iqA_\mu, \tag{41}$$

which is called the gauge covariant derivative.

Notice that we have introduced an interaction term in the Lagrangian

$$\mathcal{L}_{\text{int}} = -q\bar{\psi}\gamma^\mu \psi A_\mu = -J^\mu A_\mu, \tag{42}$$

which represents the interaction of the new A field with the charge vector current of the electron.

But what about the free-field Lagrangian for the field A? We will seek such a Lagrangian that is Lorentz invariant, and contains up to only second-order derivatives. A general such Lagrangian will contain terms of the form

$$\mathcal{L} \overset{?}{\sim} \partial_\mu A^\mu \partial_\nu A^\nu + \partial_\mu A_\nu \partial^\nu A^\mu + M A_\mu A^\nu. \tag{43}$$

Under the transformation, the A^2 terms become:

$$A_\mu A^\nu \to A_\mu A^\nu + 2A_\mu \partial^\mu \theta + \partial_\mu \theta \partial^\mu \theta. \tag{44}$$

These will not cancel in the Lagrangian, leading us to set $M = 0$.

Terms of the form

$$\begin{aligned} \partial_\mu A^\mu &\to \partial_\mu A^\mu + \partial_\mu \partial^\mu \theta \\ \partial_\mu A_\nu &\to \partial_\mu A_\nu + \partial_\mu \partial_\nu \theta. \end{aligned} \tag{45}$$

It takes only a litle playing around to notative that for the θ derivatives to cancel, only anti-symmetric combinations of the form

$$\underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu}}(\partial^\mu A^\nu - \partial^\nu A_\nu)$$

may appear, where we introduce the notation $F_{\mu\nu}$ for the antisymmetric tesnor of A derivatives. We therefore write down the Dirac Lagrangian in the form

$$\boxed{\mathcal{L}_{\text{QED}} = i\bar{\psi}\gamma^\mu \left(\partial_\mu + iqA_\mu\right)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.} \tag{46}$$

The factor of $(-1/4)$ is arbitrary for now, and chosen by convention. We remind that, as yet, the introduced q "charge" and A field are arbitrary as well – they were simply invented in order to "force" the Dirac Lagrangian to hold a given symmetry. In the next section, we will see that these can be interpreted as the electric charge, and electromagnetic four potential.

4.4.1 Dirac electrodynamics

We will now investigate the F^2 term in the Lagrangian to make sense of this new term.

First, we will consider derivatives of the tensor terms of the form

$$\partial_\lambda F_{\mu\nu} = \partial_\lambda(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_\mu\partial_\lambda A_\nu - \partial_\nu\partial_\lambda A_\mu, \tag{47}$$

where we commuted the derivatives. Noting that we may write $\partial_\lambda A_\nu = F_{\lambda\mu} + \partial_\mu A_\lambda$, we see that we can form combinations that sum to zero:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} \equiv \partial_{[\lambda} F_{\mu\nu]} = 0. \tag{48}$$

This is called the *Bianchi identity*.

For reasons that will become immediately clear, we will now fix our reference frame, and introduce notation for the derivatives of the $A^\mu \equiv (A_0, \boldsymbol{A})$ potential, $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\boldsymbol{E} \equiv -\boldsymbol{\nabla} A_0 - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{49}$$

$$\boldsymbol{B} \equiv \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{50}$$

So far, these are just arbitrary definitions, but as you can probably guess, the choices for these symbols will be clear soon.

Taking the Bianchi identity, and evaluating it with $\{\lambda, \mu, \nu\} = \{0, 1, 2\}$, and $\{\lambda, \mu, \nu\} = \{1, 2, 3\}$, we eventually arrive at

$$\partial_{[\lambda} F_{\mu\nu]} = 0 \quad \Longrightarrow \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \text{and} \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{51}$$

respectively. These are of course the Maxwell–Faraday equation, and Gauss's law for magnetism, or the source-free pair of Maxwell's equations.

Going one step further, and considering the Euler-Lagrange equations (35) with $\phi = A_\nu$, and noting that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F_{\mu\nu},$$

we find

$$\partial_\mu F^{\mu\nu} = J^\mu, \tag{52}$$

where $J^\mu \equiv q\bar{\psi}\gamma^\mu\psi$ is the Dirac vector current. Again, we fix the reference frame, and set $J^\mu = (\rho, \boldsymbol{j})$. Evaluating with $\nu = 0$ and $\nu = \{1, 2, 3\}$, results in :

$$\partial_\mu F^{\mu\nu} = J^\mu \quad \Longrightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho, \quad \text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{j} + \frac{\partial E}{\partial t}, \tag{53}$$

respectively. These are of course Gauss's law and Ampère's law; we thus recognise this as the inhomogeneous Maxwell equations.

We may therefore recognise $F^{\mu\nu}$ as the electric field strength tensor (or the Maxwell tensor), A^μ as the electromagnetic four potential, and q as the electron charge. Notice that, simply by enforcing the Dirac Lagrangian to be invariant under the U(1) symmetry, we have given rise to Maxwell's equations!

4.4.2 Charge of electrons/positrons

We previously defined the current density, $J^\mu \equiv q\bar{\psi}\gamma^\mu\psi$, which by analogy with classical electrodynamics we can recognise at the electron charge current density. We can similarly define the total current, using the plane-wave expansion (32):

$$I^\mu = \int J^\mu \mathrm{d}^3x = q\sum_s \int \frac{d^3p}{(2\pi)^3} E_p \left(b_p^s b_p^{*s} + c_p^s c_p^{*s}\right). \tag{54}$$

Note that electrons and positrons both contribute to the charge with the same sign! This means positrons have the same charge as electrons. Coupled with our result from Eq. (33), it would seem in our *classical* Dirac field theory that positrons have negative energy, and negative charge. As it will turn out in the *quantum* field theory, this interpretation becomes flipped; we will see later that positrons have positive energy, and positive charge.

4.4.3 Lorenz and Coulomb gauges

As you probably recall from classical electrodynamics, we can always make substitution $\Phi \to \Phi + \dot{\theta}$, $\boldsymbol{A} \to \boldsymbol{A} - \boldsymbol{\nabla}\theta$. This is, of course, just the statement we had in Eq. (39): $A_\mu \to A'_\mu = A_\mu + \partial_\mu \theta$. The choice of θ function is called a gauge choice. While any choice is valid, some choices turn out to be more convenient. A common example is the *Lorenz* gauge. We consider $\partial^\mu A_\mu \; \partial^2 \theta = 0$, and choose solutions such that

$$\partial^\mu A_\mu = 0 \qquad \text{Lorenz gauge.} \tag{55}$$

This choice is widely used, and is convenient due to the explicit Lorentz invarience.

Another common choice is the Coulomb gauge, which is explicitly not Lorentz invariant. In the Coulomb gauge we assert that both parts of the derivative are zero:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0\,, \quad \dot{\Phi} = 0 \qquad \text{Coulomb gauge.} \tag{56}$$

In the Lorenz gauge $\partial^\mu A_\mu = 0$, the Euler-Lagrange equations for the free photon field become particularly simple. The resulting equation for the A^μ

$$\partial^2 A^\mu = 0, \tag{57}$$

is just the massless KG equation (for each for the four A^μ components). Thus, we can expand the solution in terms of KG plane wave solutions:

$$A_\mu(x) = \sum_s \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{\sqrt{2\mathcal{E}_k}} \epsilon^s_\mu \left(a^s_k e^{-ikx} + \hat{a}^{s*}_k e^{ikx} \right), \tag{58}$$

where ϵ is the polarisation unit vector, and a is a complex amplitude. The difference from the general KG case, however, is that there are only two independent polarisation vectors

$$\epsilon^s_\mu k^\mu = 0,$$

which is a result of the lack of an $m_\gamma A^\mu A_\mu$ term in the Lagrangian. For example, for $\boldsymbol{k} = k_z \boldsymbol{e}_z$, two modes may be taken as $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$.

4.5 Non-relativistic limit, electron g-factor

It's extremely instructive to consider the non-relativistic limit of the Dirac equation. Firstly, if we are inventing a "new" relativistic theory, as good scientists, we must ensure that we do not contradict any existing experimental evidence. The simplest way to do this is to show that, in the non-relativistic limit, we recover the existing non-relativistic theory which is already well tested and known to be accurate in the non-relativistic regime. Secondly, it will make it clear why we have term two of the degrees of freedom "spin".

As we saw above, in the presence of a vector potential, $A^\mu = (V, \boldsymbol{A})$, the Dirac equation is:

$$i\gamma^\mu(\partial_\mu + iqA_\mu)\psi - m\psi = 0, \tag{59}$$

where $q = -e = -|e|$ is the electron charge. For this exercise, it is much easier if we work within the Dirac basis instead of the Weyl basis. We first write the Dirac solution in the form

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} e^{-imt}, \tag{60}$$

where ϕ and χ are both two-component spinors (and we have taken $p^\mu x_0 \mu \approx mt$, from the non-relativistic limit). The equation becomes

$$\begin{pmatrix} i\partial_t - qV - m & 0 \\ 0 & i\partial_t - qV + m \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} e^{-imt} - i \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot(i\boldsymbol{\nabla} + q\boldsymbol{A}) \\ \boldsymbol{\sigma}\cdot(i\boldsymbol{\nabla} + q\boldsymbol{A}) & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} e^{-imt} = 0, \tag{61}$$

which, after propagating through the time derivatives, leads to a pair of coupled equations:

$$(i\partial_t - qV)\phi - \boldsymbol{\sigma}\cdot(i\boldsymbol{\nabla} - q\boldsymbol{A})\chi = 0 \tag{62}$$

$$(i\partial_t - qV + 2m)\chi - \boldsymbol{\sigma}\cdot(i\boldsymbol{\nabla} - q\boldsymbol{A})\phi = 0. \tag{63}$$

In the non-relativistic case, the "$2m$" term on the left-hand-side of the second equation should dominate, allowing us to approximate

$$\chi \approx \frac{1}{2m}\boldsymbol{\sigma}\cdot(i\boldsymbol{\nabla} - q\boldsymbol{A})\phi. \tag{64}$$

It is this common to call ϕ the "large" Dirac component (in non-relativistic problems), and χ the "small" component. Had we used the Weyl basis, it would instead be $\psi_L \pm \psi_R$ as the large/small components.

Plugging this approximation back into the first equation, we arrive at:

$$\left(\frac{(\boldsymbol{p} - q\boldsymbol{A})^2}{2m} + qV - \boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})\frac{q}{2m}\right)\phi = i\partial_t\phi, \tag{65}$$

where we used the identity:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b} + i\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b}).$$

We can see that this is exactly the expression we would expect for the interaction of an electron with electric (V) and magnetic ($\boldsymbol{A}$) potentials. What's even more amazing, is that if we notice the spin vector for an electron is $\boldsymbol{s} = (1/2)\boldsymbol{\sigma}$, and that $\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{B}$, then we have a term equal to the interaction of spin wth magnetic field:

$$\frac{-q}{m}(\boldsymbol{s}\cdot\boldsymbol{B}) = -g_s\mu_B(\boldsymbol{s}\cdot\boldsymbol{B}),$$

where $\mu_B = |e|/(2m)$ is the Bohr magneton, and g_s is the gyromagnetic ratio, or g-factor. Since the electron has spin $\hbar/2$, this shows us that $g_s = 2$. It's worth taking a moment to reflect on this result. Not only has the Dirac equation led naturally to spin interactions, but it has predicted the correct[2] electron g-factor of 2!

4.6 Introduction to quantum electrodynamics

Roughly speaking, in going from classical to quantum mechanics, we "promoted" the momentum/energy variables to *operators*. Recall for a scalar field that satisfies the KG equation, with $\mathcal{L} \sim \partial_\mu\phi\partial^\mu\phi$, we "quantise" the field by promoting the amplitudes to operators:

$$\hat{\phi} = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2\mathcal{E}_k}}\left(\hat{a}_{\boldsymbol{k}}e^{-ikx} + \hat{a}^\dagger_{\boldsymbol{k}}e^{ikx}\right), \tag{66}$$

where the creation ($a^\dagger$) and annihilation (a) operators obey the commutation relation

$$[a_k,\, a^\dagger_{k'}] = (2\pi)^3\delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}'). \tag{67}$$

In a similar fashion, we quantise the Dirac field by expanding in plane wave solutions, and "promoting" the amplitudes to creation/annihilation operators:

$$\hat{\psi}(x) = \sum_s \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\mathcal{E}_p}}\left[\hat{b}^s_p u^s(p)e^{-ipx} + \hat{c}^{\dagger s}_p v^s(p)e^{ipx}\right], \tag{68}$$

[2] As it turns out, this is also just an approximation. The electron g-factor differs from exactly two: $(g-2)/2 \approx 1.159652... \times 10^{-3}$. This result is explained by a full quantum field theory, which accurately predicts the correct experimental result to more than 10 digits (!), making this the most accurately verified prediction in all of physics.

and similarly for $\bar{\psi}$, where the b and c operators obey *anti*-commutator relations:

$$\{b^s_{\boldsymbol{p}},\, b^{\dagger r}_{\boldsymbol{q}}\} = \{c^s_{\boldsymbol{p}},\, c^{\dagger r}_{\boldsymbol{q}}\} = (2\pi)^3\delta_{rs}\delta(\boldsymbol{p}-\boldsymbol{q}) \tag{69}$$

(for all other combinations $\{\} = 0$).

For the photon field, it becomes simplest if we work in the Lorenz gauge, where $\partial_\mu A^\mu = 0$. In this case, the photon potential is seen to satisfy the KG equation, meaning we can expand it over plane wave solutions. The difference from the general KG case, however, is that there are only two independent polarisation vectors

$$\epsilon^s_\mu k^\mu = 0,$$

which is a result of the lack of an $m_\gamma A^\mu A_\mu$ term in the Lagrangian. For example, for $\boldsymbol{k} = k_z\boldsymbol{e}_z$, two modes may be taken as $(0,1,0,0)$ and $(0,0,1,0)$.

$$\hat{A}_\mu(x) = \sum_s \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2\mathcal{E}_k}}\epsilon^s_\mu\left(\hat{a}_k e^{-ikx} + \hat{a}^\dagger_k e^{ikx}\right), \tag{70}$$

Notice that the commutation relations encode the spin-statistics properties. For Fermions

$$|p\rangle \propto b^\dagger_p|0\rangle \tag{71}$$

$$\begin{aligned}|p,q\rangle &\propto b^\dagger_q b^\dagger_p|0\rangle\\ |q,p\rangle &\propto b^\dagger_p b^\dagger_q|0\rangle\\ &= -|p,q\rangle,\end{aligned} \tag{72}$$

since $\{b^\dagger_p,\, b^\dagger_q\} = 0$. This encodes the Fermion antisymmetry.

With the correct normalisation, one-particle states (denoted by momentum p and spin s) are written,

$$|\boldsymbol{p}, s\rangle = \sqrt{2E_p}\, b^{\dagger s}_p|0\rangle. \tag{73}$$

We may write the Hamiltonian in this form too:

$$H_{\text{Dirac}} = \sum_s \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} E_{\boldsymbol{p}}\left(b^{s\,\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}} + c^{s\,\dagger}_{\boldsymbol{p}} c^s_{\boldsymbol{p}}\right) \tag{74}$$

4.7 Perturbation theory and Feynman rules

We begin by separating the QED Lagrangian into parts involving only the free Dirac terms, the free Maxwell terms, and an interaction term:

$$\mathcal{L}_{\text{QED}} = i\bar{\psi}\left(\not{\partial} + iq\not{A}\right)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{75}$$

$$= \mathcal{L}_{\text{Dirac}}(\psi) + \mathcal{L}_{\text{Maxwell}}(A) + \mathcal{L}_{\text{int}}(\psi, A), \tag{76}$$

where

$$\mathcal{L}_{\text{int}} = -qA_\mu\bar{\psi}\gamma^\mu\psi. \tag{77}$$

The core assumption is that these interaction terms are small, and can be treated perturbatively. It turns out that in QED this is generally a good assumption; there are some subtleties here that I will leave up to the textbooks to discuss. We will take this as an underlying assumption, and just move forwards.

We note that we have $q = -e = -|e|$ is the electron charge. In our units, this is $e = \sqrt{4\pi\alpha}$, where $\alpha \approx 1/137$ is the fine structure constant.

We recall that in perturbation theory, we calculate matrix elements of the S matrix. We have:

$$|\psi(t)\rangle = \hat{U}(t)|\psi(t=0)\rangle, \tag{78}$$

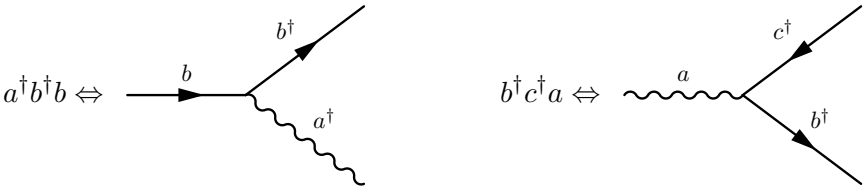
where $U(t) = e^{-iHt}$ is the time evolution operator, and H is the Hamiltonian. More concretely, we have

$$U = \exp\left(-i\int_{t_0}^{t} H_{\text{int}}(t')\mathrm{d}t'\right), \tag{79}$$

which we expand perturbatively (in H_{int}. The S-matrix is defined $S = \langle f|U|i\rangle$ in the limit that $t_0 \to -\infty$, $t \to \infty$. We have

$$H_{\text{int}} \sim \int -qA_\mu \bar{\psi}\gamma^\mu\psi. \tag{80}$$

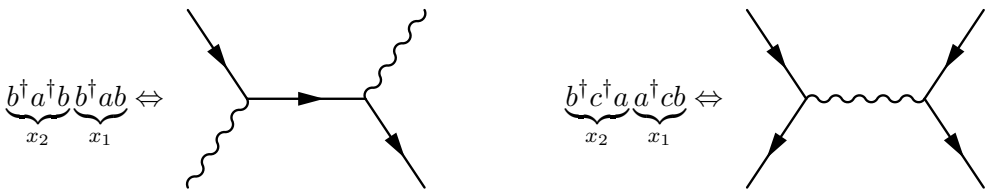
Without going through all the details, notice that the A^μ term contributes terms of the form $\sim (a + a^\dagger)$, while ψ and $\bar{\psi}$ contribute terms of the form $\sim (b + c^\dagger)$ and $\sim (c + b^\dagger)$, respectively. Therefore, to first-order in perturbation theory, eight combinations of three field operators may appear. Notice that, since each of the terms in the interaction Hamiltonian conserve charge (and lepton number), each of these combinations will also conserve charge and lepton number. There is no conservation of photon number (or indeed, overall particle number). We may associate these with parts of diagrams, known as Feynman diagrams:[3]



(81)

These two particular terms can be understood as the destruction of an electron and the creation of an electron and a photon (left), and the destruction of a photon and the creation of an electron and a positron (right). Note that, at first-order, all terms are evaluated at the same space-time location, known as a "vertex" – in the diagrams, this is the location of intersection of all lines. Notice that each vetex will always be some combination of two Fermion operators and a photon operator (the only combinations that exist). Each of these vertices will apear in the calculation of the S-matrix with a factor $iq\gamma^\mu$, coming directly from Eq. (79). Each of the lines in the diagrams also correspond to the appearance of one of the u or v spinors, in the case of Fermion lines, or an ϵ polarisation vector for a photon line.

To second order, there are many more terms. At second order, there will now appear two vertices, each evaluated at a different location. Notice, however, that since $a|0\rangle = 0$, only terms with that have a combination of the form $aa^\dagger$ (or $bb^\dagger$) survive. The terms of this form span the two vertices, and are evaluated at different locations. These terms are known as *propagators*, since roughly speaking, they are proportional to the probability of propagating from one location to another. At second order, diagrams of the form:



(82)

appear (along with many other combinations).

The association between terms in the perturbation theory expansion and parts of diagrams give a particularly nice way to compute relevant amplitudes. A more formal treatment of the very hand-wavy description above leads to a set of rules, known as Feynman rules, which link diagrams (representing perturbation theory contributions to some amplitude) to the integral representing that amplitude. We will not formally derive or prove the rules here, for that, see one of the textbooks above. Instead, we shall just state them in the next section.

[3] Note: in my diagram I draw time going from left to right, because everyone knows that's the way time flows. Many other textbooks have time going upwards, since that's the common way to draw space-time diagrams. The amplitudes are symmetric, but the interpretation is different, so just be careful comparing different sources.

4.7.1 Feynman rules for QED

External Lines: For each external line, we include a factor of:

- $\implies \bar{u}_s$ (outgoing Fermion)
- $\implies u_s$ (incoming Fermion)
- $\implies v_s$ (outgoing anti-Fermion)
- $\implies \bar{v}_s$ (incoming anti-Fermion)
- $\implies \epsilon_s^{\mu *}$ (outgoing photon)
- $\implies \epsilon_s^{\mu}$ (incoming photon)

Vertex: For each vertex, we ascociate the factor

$$ig_e\gamma^\mu(2\pi)^4\delta^{(4)}(\sum[p_{in}-p_{out}])$$

Internal lines (propagator): For an internal Fermion line, the propagator is

$$i\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\not{q}+m}{q^2-m^2}, \tag{83}$$

and for an internal photon line, the propagator is

$$-i\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{g_{\mu\nu}}{q^2}. \tag{84}$$

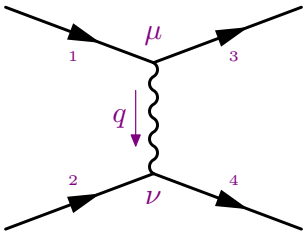
Symmetrisation: Finally, we must include a relative negative sign between any pair of diagrams that differ only by a Fermion exchange.

Amplitude: After putting everything together, you are left with

$$-i(2\pi)^4\delta(\Delta p)\mathcal{M}, \tag{85}$$

where $\mathcal{M}$ is the amplitude.

As a simple example, we will consider the simple Coulomb scattering example. We won't completely evaluate the diagram, but just write down the initial expression.



We do not normally need to label the vertices – the purple labels are just instructive for our first example. The direction (sign) of internal momenta are arbitrary (they will be integrated out), but must be consistent if they appear in more than one term. Here, I will use a short-hand, where u_1 means $u_{s1}(p_1)$.

$$\begin{aligned}\bar{u}_3(ig_e)\gamma^\mu u_1(2\pi)^4\delta(p_1-p_3-q)\\ \times\ -i\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{g_{\mu\nu}}{q^2}\\ \times\ \bar{u}_4(ig_e)\gamma^\nu u_2(2\pi)^4\delta(p_2-p_4+q).\end{aligned} \tag{86}$$

Integration over q is taken care of simply due to one of the delta functions. We'll choose the first one, which implies $q=p_1-p_3$. Then, we are left with:

$$(-i)\underbrace{(-|e|^2)\frac{(\bar{u}_3\gamma^\mu u_1)(\bar{u}_4\gamma_\nu u_2)}{(p_1-p_3)^2}}_{\mathcal{M}}(2\pi)^4\delta(p_2-p_4+p_1-p_3). \tag{87}$$

As an instructive example, let's consider the non-relativistic limit of this amplitude. In this case, we may approximate $p^\mu = (m, \boldsymbol{p})$, with the mass term dominating. This means we can replace $\bar{u}\gamma^\mu u$ with $\bar{u}\gamma^0 u = u^\dagger u$. Also, note that $p_1^\mu - p_3^\mu \approx (\Delta p^2/2m, \Delta \boldsymbol{p})$ with $|m| \gg |\boldsymbol{p}|$ (elastic approximation). Therefore, with $\boldsymbol{q} = \boldsymbol{p_1} - \boldsymbol{p_3}$, we have:

$$-e^2 \frac{(u^\dagger u)(u^\dagger u)}{-|\boldsymbol{q}|^2}. \tag{88}$$

Taking the Fourier transform of the interaction part:

$$V(\boldsymbol{r}) = e^2 \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{|\boldsymbol{q}|^2} e^{-\boldsymbol{q}\cdot\boldsymbol{r}} \tag{89}$$

$$= \frac{e^2}{4\pi r}, \tag{90}$$

which is of course the expected classical form for the Coulomb interaction potential.

4.7.2 Calculation of observables

An important reminder, that in a calculation of any relevant property, *all* Feynman diagrams (up to the given order of perturbation theory) that have indistinguishable initial and final states must be taken into account. The total amplitude is then the sum of all individual amplitudes:

$$\mathcal{M}_{\text{Tot.}} = \sum_i^{\text{diagrams}} \mathcal{M}_i. \tag{91}$$

Then

$$\text{Observable} \propto |\mathcal{M}_{\text{Tot.}}|^2 \tag{92}$$

For example, in the above Coulomb example, there is a second diagram that looks very but has the '3' and '4' external lines swapped. This diagram enters with a relative -1 factor due to the exchange symmetry.

In most cases, there are at least several diagrams that contribute. Calculating $|\mathcal{M}|^2 = \mathcal{M}^\dagger \mathcal{M}$ directly quickly becomes quite daunting. There are several 'tricks' and relations we can take advantage of, however, that will greatly simplify this.

Making use of the spin-completeness relations, Eq. (31), will result in a need to calculate traces of matrix elements. Be very careful with the re-ordering of terms; there are many non-commuting matrices involved. It is sometimes helpful to write explicitly the matrix indexes

$$\gamma^\mu \to \gamma^\mu_{ij},$$

to help avoid any confusion (each of the four $\gamma^{0,1,2,3}$ is itself a 4x4 matrix, with indices i, j; note that we reserve Greek indices for space-time indices, and use Latin characters for spinor indices, even though they range 1-4).

Consider the example

$$\sum_{s,r} [\bar{u}^s(p)\gamma^\mu u^r(q)][\bar{u}^r(q)\gamma^\nu u^s(p)] = \sum_{s,r} [\bar{u}^s_a(p)\gamma^\mu_{ab} u^r_b(q)][\bar{u}^r_c(q)\gamma^\nu_{cd} u^s_d(p)] \tag{93}$$

$$= \left(\sum_s \bar{u}^s_a(p) u^s_d(p)\right) \gamma^\mu_{ab} \left(\sum_r u^r_b(q)\bar{u}^r_c(q)\right) \gamma^\nu_{cd} \tag{94}$$

$$= (\not{p} + m)_{da}\, \gamma^\mu_{ab}\, (\not{q} + m)_{bc}\, \gamma^\nu_{cd}, \tag{95}$$

where we made use of the spin-completeness relations, Eq. (31). Note that this is in the form of a matrix equation, which is by definition, a trace:

$$A_{da}B_{ab}C_{bc}D_{cd} = A_{da}B_{ab}E_{bd} = A_{da}F_{ad} = G_{dd} \equiv \mathrm{tr}(G) = \mathrm{tr}(ABCD). \tag{96}$$

Therefore:

$$\sum_{s,r}[\bar{u}^s(p)\gamma^\mu u^r(q)][\bar{u}^r(q)\gamma^\nu u^s(p)] = \mathrm{tr}\left[(\not{p}+m)\gamma^\mu(\not{q}+m)\gamma^\nu\right], \tag{97}$$

These terms can be evaluated by making use of several trace theorems. There are many theorems, we'll just go over the simplest of them here. For example, the trace of any product of an odd number of γ matrices is zero:

$$\mathrm{tr}(\overbrace{\gamma^\mu\gamma^\nu\ldots\gamma^\lambda}^{\text{Odd \#}}) = 0. \tag{98}$$

To prove this, take advantage of $(\gamma^5)^2 = 1$ and :

$$\begin{aligned}
\mathrm{tr}(\gamma^\mu) &= \mathrm{tr}(\gamma^5\gamma^5\gamma^\mu) && (99)\\
&= -\mathrm{tr}(\gamma^5\gamma^\mu\gamma^5) && (100)\\
&= -\mathrm{tr}(\gamma^5\gamma^5\gamma^\mu) && (101)\\
&= -\mathrm{tr}(\gamma^\mu) && (102)
\end{aligned}$$

where in the second line we used the γ matrix identity $\{\gamma^\mu, \gamma^5\} = 0$, and in the third line, we used the cyclic property trace identity $\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB)$. Since $\mathrm{tr}(\gamma^\mu) = -\mathrm{tr}(\gamma^\mu)$, we must have $\mathrm{tr}(\gamma^\mu) = 0$. The exact same logic works for any odd number of γ matrices.

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}. \tag{103}$$

It is often the case that the polarisation information of the incoming/outgoing particles is not known. In such cases, we must *average* over the spins in the initial state, and *sum* over spins in the final state. For example, in the common situation where there are two incoming particles (i.e., two particles in the initial state), with any number of particles in the final state, we have:

$$\left(\frac{1}{4}\sum_{s_1,s_2}\right)\sum_{s_3,s_4,\ldots}|\mathcal{M}|^2. \tag{104}$$

As a final note, it is often the case that we will be calculating scattering cross-sections or decay rates. A cross-section is a measure of how strongly a particular scattering process occurs

The scattering rate (number of scattering events per unit time) will be proportional to the number of target particles in the interaction region, N_{Target}. It will also be proportional to the number of incoming scattering particles passing through the interaction region each unit of time, which is $n\,v$, where n is the incoming particle volume density, and v is velocity. It will also, of course, depend on the details of the particle physics interactions. From dimensionality arguments, the combination of remaining terms must have dimension of area – we call it the cross section, σ:

$$R = N_{\text{Target}} \times n_{\text{incoming}}\, v \times \sigma. \tag{105}$$

We can think of this equation as the *definition* of σ. In the classical "hard-sphere" analogue, it is clear why the term cross-section is used. It is often the case that we are actually interested in the differential cross section, as a function of energy/momentum/angle or whatever.

Without proof, I give the formula for differential cross-section (per solid angle), for the typical case of scattering between two particles (in their centre of mass frame):

$$\sigma = S\int|\mathcal{M}|^2(2\pi)^4\delta(\Delta p)\frac{1}{4\mathcal{E}_1\mathcal{E}_2 v}\prod_i^{\text{out}}\left(\frac{\mathrm{d}^3p_i}{(2\pi)^3}\frac{1}{2\mathcal{E}_i}\right) \tag{106}$$

where $v = |\vec{v}_1 - \vec{v}_2|$. It is common to re-write the denominator in Lorentz invariant form:

$$\mathcal{E}_1\mathcal{E}_2 v = \sqrt{(p_1\cdot p_2)^2 - (m_1m_2)^2}.$$

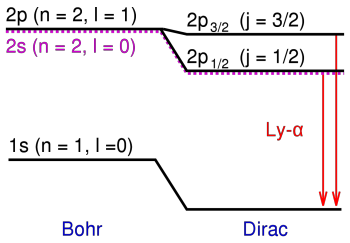


Figure 1: Fine structure of Hydrogen.[4]

Similarly, for the decay of a single particle at rest, we have:

$$\Gamma = \frac{1}{2m}|\mathcal{M}|^2(2\pi)^4\delta(\Delta p)\prod_f\left(\frac{\mathrm{d}^3p_f}{(2\pi)^3}\frac{1}{2\mathcal{E}_i}\right). \tag{107}$$

In the common case of just two particles in the final state, we can write

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}_{\text{c.o.m}} = \frac{|\boldsymbol{p}_4|}{4\mathcal{E}_1\mathcal{E}_2 v}\frac{1}{(2\pi)^2 4E_{\text{com}}}|\mathcal{M}|^2, \tag{108}$$

where $E_{\text{com}} = \mathcal{E}_1 + \mathcal{E}_2$ is the centre of mass energy.

4.8 Bethe's treatment of the Lamb shift

In the purely non-relativistic picture of a hydrogen-like ion (Z protons, 1 electron), the energy levels depended only on the principle quantum number, n:

$$\varepsilon(n,l,m) = \varepsilon(n) = -\frac{Z^2}{n^2}R_y, \tag{109}$$

where

$$R_y = \frac{1}{2}\alpha^2(m_ec^2) \approx 13.61\,\text{eV}$$

is the Rydberg constant. Dirac's theory of the electron accurately described the appearance of the so-called *fine structure* splitting, the difference in energy between states of different total angular momentum, $j = l \pm 1/2$:

$$\Delta E_{\text{fs}} \sim \alpha^4(m_ec^2). \tag{110}$$

According to the Dirac equation, the hydrogen spectrum should depend *only* on j (and n), but not l [Fig. 1]. However, by the late 1940s, it had been observed this was not the case. An extremely small shift on the order $\sim \alpha^5(m_ec^2)$ had been observed between the $2s_{1/2}$ and $2p_{1/2}$ states of hydrogen by Lamb and Retherford[5]:

$$\Delta E_{2s_{1/2}-2p_{1/2}} \simeq 1000\,\text{MHz} \simeq 5\times10^{-6}\,\text{eV}, \tag{111}$$

which has since been measured to 1057.845(3) MHz (with the s state at a *higher* energy than the p state).

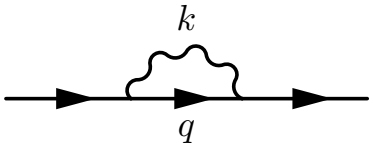
The Lamb shift was first explained theoretically by Hans Bethe, in what was also one of the first successful calculations using the techniques from quantum electrodynamics[6]. It was also the first calculation that managed to remove the "infinities" the crop up in QED calculations; a process now known as renormalisation. As such, this was among the most influential calculations of modern physics.

The shift was postulated to be explained by the so-called electron "self-energy", the name given for process whose Feynman diagram is:

[4] Image: Wikipedia (Tobix10), public domain.

[5] W. E. Lamb and R. C. Retherford, "Fine structure of the hydrogen atom by a microwave method", Phys. Rev. **72**, 241–243 (1947).

[6] H. A. Bethe, "The Electromagnetic Shift of Energy Levels", Phys. Rev. **72**, 339–341 (1947).



This diagram involves two internal lines (two propagators), and is thus rather difficult to compute directly (it is done in the Sakurai textbook[7]). Instead, we will follow the semi-relativistic approach of Bethe, in which the electron wavefunction is treated non-relativistically.

In the Coulomb gauge, we choose $A_0 = 0$, and write an interaction Hamiltonian from the Lagrangian (77)

$$\hat{h}_{\text{int}} = -q\gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{A}. \tag{112}$$

In the non-relativistic limit, and keeping terms only up to linear in A, we have

$$\hat{h}_{\text{int}} \approx -\frac{q}{m} \boldsymbol{p} \cdot \boldsymbol{A}, \tag{113}$$

which can be seen from Eq. (65). From regular quantum mechanics, we expect the energy shift to be

$$\delta\varepsilon_a = \sum_n \frac{\langle a|h|n\rangle\langle n|h|a\rangle}{\varepsilon_a - \varepsilon_n}, \tag{114}$$

where $\boldsymbol{A}$ would be interpreted as a classical external field. In our case, however, there is no classical field, and we instead recognise $\boldsymbol{A}$ as the photon operator, Eq. (70). The other change we must make is to consider the photon number as part of the initial and intermediate wavefunctions:

$$|a\rangle \to |a, 0\rangle \qquad \text{(electron in state } a\text{; 0 photons)} \tag{115}$$

$$|n\rangle \to |n, 1\rangle \qquad \text{(electron in state } n\text{; 1 photon)}. \tag{116}$$

We will write the photon energy $E_k = \omega$, so we also have $\varepsilon_n \to \varepsilon_n + \omega$. Putting it together, the energy shift can be expressed as:

$$\delta\varepsilon_a = \frac{e^2}{m^2} \sum_n \sum_{s=\pm} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2\omega} \frac{|\langle a, 0|\boldsymbol{p} \cdot \boldsymbol{\epsilon}_s (a_k^\dagger e^{ikx} + a_k e^{-ikx})|n, 1\rangle|^2}{\varepsilon_a - \varepsilon_n - \omega}. \tag{117}$$

Keep in mind the a,n states are bound atomic states, not free electron states.

We first make the dipole approximation $kx \approx \omega t$. This is essentially the statement that the electric field of the photon is roughly constant across the extent of the atom, which is valid for photons with wavelength $\lambda \gg a_0$; in other words $c\hbar/\omega \gg \hbar/(m_e c\alpha)$, or $\omega \ll m_e c^2 \alpha \sim 10^4\,\text{eV}$.[8] In this case, there is no spatial dependence in the photon part of the wavefunction. Further, the a,$a^\dagger$ terms act only on $|0\rangle, |1\rangle$. This means we can separate the photon and electron parts.

For the photon part, we have:

$$|\langle 0| \overbrace{a_k^\dagger e^{ikx} + a_k e^{-ikx}}^{y} |1\rangle|^2 = \langle 0|y|1\rangle\langle 1|y^\dagger|0\rangle \tag{118}$$

$$= \sum_m \langle 0|y|m\rangle\langle m|y^\dagger|0\rangle \tag{119}$$

$$= \langle 0|yy^\dagger|0\rangle = 1, \tag{120}$$

where we used a trick in the second line, noting that only the $m = 1$ non-zero, due to only single $a/a^\dagger$ operator, and then used the closure relation to get to the final line, and then made use of:

$$\begin{aligned} a^\dagger|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ a|m\rangle &= \sqrt{m}\,|m-1\rangle. \end{aligned} \tag{121}$$

[7] J. J. Sakurai, *Advanced Quantum Mechanics* (Cambridge University Press, 1967).

[8] It's not clear this is valid; after all, who knows what the frequency of the external photon is? But we can always consider higher-order terms if need be, this is a reasonable starting point.

For the electron part, we can make use of the spherical symmetry, and write $\mathrm{d}^3k = k^2\mathrm{d}|\boldsymbol{k}|\mathrm{d}\Omega = \omega^2\mathrm{d}\omega\mathrm{d}\Omega$. For the sum over polarisations and integration over angles, we can use a well known result from the dipole approximation:

$$\sum_{s=\pm}\int |\langle a|\boldsymbol{p}\cdot\boldsymbol{\epsilon}_s|n\rangle|^2\,\mathrm{d}\Omega = 2\left(\frac{4\pi}{3}\right)|\langle a|\boldsymbol{p}|n\rangle|^2. \tag{122}$$

We won't prove this directly here, though the result is fairly intuitive. Since there is no preferred direction, the integral is simply the average over the three independent directions for $\boldsymbol{\epsilon}$, hence the $4\pi/3$ term. The factor of 2 comes from the two spin states.

The final expression we have is then:

$$\delta\varepsilon_a = \frac{2\alpha}{3\pi}\frac{1}{m^2}\sum_n |\langle a|\boldsymbol{p}|n\rangle|^2\int_0^\infty \frac{\omega\,\mathrm{d}\omega}{\varepsilon_a-\varepsilon_n-\omega}, \tag{123}$$

where we replaced $e^2 = 4\pi\alpha$. Those with a keen eye will have spotted the enormous problem immediately. This integral does not converge; the energy correction is infinite! This is the classic problem in any quantum electrodynamics calculation involving closed loops. We may argue that it's not physical for the energy integral to extend all the way to infinity; after all, this physically means the energy of the photon in the intermediate state. How could such a photon have an energy larger than that of the electron which emitted it, so perhaps we can introduce a cut-off $\infty \to \omega_{\mathrm{cut-off}} \approx m_e c^2 \approx$ MeV? Of course, this argument is weak. What's worse, is that the linear divergence depends very strongly on the cut-off.

To investigate the issue, we consider the exact same correction, but to a *free* electron (rather than a bound atomic electron), with $|a\rangle \to |p\rangle$. Almost everything is the same, except that the free electron is a *momentum* eigenstate. This means there are only diagonal $\langle p|\boldsymbol{p}|p'\rangle$ matrix elements, so only $|n\rangle = |p\rangle$ survives in the sum. The correction to the energy of the free electron is thus:

$$\delta\varepsilon_p^{\mathrm{free}} = -\frac{2\alpha}{3\pi}\frac{1}{m^2}|\langle p|\boldsymbol{p}|p\rangle|^2\int_0^{\omega_{\mathrm{cut}}}\mathrm{d}\omega = -\frac{4\alpha}{3\pi}\frac{\omega_{\mathrm{cut}}}{m}\frac{|\langle\boldsymbol{p}\rangle|^2}{2m}, \tag{124}$$

which also diverges.

But hang on! This is simply a correction to the kinetic energy of a momentum eigenstate,

$$\varepsilon_p = \frac{p^2}{2m}\left(1-\frac{4\alpha}{3\pi}\frac{\omega_{\mathrm{cut}}}{m}\right), \tag{125}$$

which certainly shouldn't exist! After all, the *definition* of kinetic energy (or momentum, or mass) is simply $\varepsilon = p^2/2m$ (as a non-relativistic approximation from $\mathcal{E}^2 = p^2 + m^2$). The interpretation is that the mass and coupling terms in the QED Lagrangian, which we called m, e, are *not* necessarily the observed quantities. If we call the "bare" mass, that appears in the QED Lagrangian m_{bare}, then we should make the connection to the real, observed mass:

$$\frac{p^2}{2m_{\mathrm{obs}}} \equiv \frac{p^2}{2m_{\mathrm{bare}}}(1-\frac{4\alpha}{3\pi}\frac{\omega_{\mathrm{cut}}}{m}). \tag{126}$$

We can interpret this as a QED radiative correction to the bare mass, which, by observation, must be equal to the observed mass:

$$m_{\mathrm{obs}} \approx m_{\mathrm{bare}}\left(1+\frac{4\alpha}{3\pi}\frac{\omega_{\mathrm{max}}}{m}\right) \tag{127}$$

(where I kept terms only up to first-order in α).

When we calculated the energy levels of hydrogen (without the QED correction), we solved the Schrodinger equation:

$$H = \frac{p^2}{2m} + V,$$

and of course used the *observed* mass. Therefore, when estimating energy corrections, we must subtract off the part of the energy correction coming from the mass correction to $p^2/2m$, since it was already included in

calculation of the unperturbed energy levels. (This is equivalent to solving the problem from the beginning, and then absorbing the divergent mass correction into the definition of $m_{\rm obs}$) This is the general idea of *mass renormalisation*, which you'll study more formally later.

We must evaluate the observable energy correction as

$$\delta\varepsilon_a^{\rm obs} = \delta\varepsilon_a^{\rm QED} - \langle a|\frac{p^2}{2m_{\rm obs}} - \frac{p^2}{2m_{\rm bare}}|a\rangle \tag{128}$$

$$= \delta\varepsilon_a^{\rm QED} + \langle a|p^2|a\rangle \frac{2\alpha}{3\pi\, m_{\rm obs}^2}\int_0^{\omega_{\rm cut}} {\rm d}\omega \tag{129}$$

which can be seen from Eq. (126). That is, the *observable* self-energy shift is the *difference* between the bound-electron self-energy and the free electron self-energy. Putting the terms together, we have:

$$\delta\varepsilon_a^{\rm obs} = \frac{2\alpha}{3\pi}\frac{1}{m^2}\int_0^{\omega_{\rm cut}} \left(\sum_n \frac{\omega\,|\langle a|\boldsymbol{p}|n\rangle|^2}{\varepsilon_a - \varepsilon_n - \omega} + \langle a|p^2|a\rangle\right) {\rm d}\omega, \tag{130}$$

(we re-labelled $m_{\rm obs} \to m$). To evaluate this, we make use of the "closure" trick:

$$\langle a|p^2|a\rangle \equiv \langle a|\boldsymbol{p}\cdot\boldsymbol{p}|a\rangle = \langle a|\boldsymbol{p}\cdot\overbrace{\Big(\sum_n |n\rangle\langle n|\Big)}^{=1}\boldsymbol{p}|a\rangle \tag{131}$$

$$= \sum_n |\langle a|\boldsymbol{p}|n\rangle|^2, \tag{132}$$

which allows us to combine the terms:

$$\delta\varepsilon_a^{\rm obs} = \frac{2\alpha}{3\pi}\frac{1}{m^2}\sum_n |\langle a|\boldsymbol{p}|n\rangle|^2(\varepsilon_n - \varepsilon_a)\int_0^{\omega_{\rm cut}} \frac{1}{\varepsilon_n - \varepsilon_a + \omega}\,{\rm d}\omega \tag{133}$$

$$= \frac{2\alpha}{3\pi}\frac{1}{m^2}\sum_n |\langle a|\boldsymbol{p}|n\rangle|^2(\varepsilon_n - \varepsilon_a)\ln\left(\frac{\varepsilon_n - \varepsilon_a + \omega_{\rm cut}}{\varepsilon_n - \varepsilon_a}\right). \tag{134}$$

While we still have a divergence, it's now only a *logarithmic* divergence. The answer will depend only very weakly on the chosen cut-off. We'll come back to this point in a moment.

Now all that remains is to evaluate the actual correction, and compare to experiment. For this, we will use the "large log" approximation. Note that $(\varepsilon_n - \varepsilon_a) \sim R_y \sim 10\,{\rm eV}$, the typical energy differences between atomic states. Whereas $\omega_{\rm cut} \sim mc^2 \sim 10^6\,{\rm eV}$ is much larger. Therefore, we can make the approximation that the logarithm is independent of n, and replace $(\varepsilon_n - \varepsilon_a) \approx \Delta_{\rm avg.}$:

$$\delta\varepsilon_a^{\rm obs} \approx \frac{2\alpha}{3\pi}\frac{1}{m^2}\left[\sum_n |\langle a|\boldsymbol{p}|n\rangle|^2(\varepsilon_n - \varepsilon_a)\right]\ln\left(\frac{\omega_{\rm cut}}{\Delta_{\rm avg.}}\right) \tag{135}$$

Then, the only part left to do is evaluate the electron matrix element. We will use a very elegant trick to do this, first employing the closure relation, then using a commutator with the Hamiltonian:

$$\sum_n |\langle a|\boldsymbol{p}|n\rangle|^2(\varepsilon_n - \varepsilon_a) = \sum_n \langle a|\boldsymbol{p}|n\rangle\langle n|\boldsymbol{p}|a\rangle(\varepsilon_n - \varepsilon_a) \tag{136}$$

$$= \sum_n \langle a|\boldsymbol{p}|n\rangle\langle n|[H,\boldsymbol{p}]|a\rangle \tag{137}$$

$$= \sum_n \langle a|\boldsymbol{p}\cdot[H,\boldsymbol{p}]|a\rangle = \frac{1}{2}\sum_n \langle a|\Big[\boldsymbol{p},[H,\boldsymbol{p}]\Big]|a\rangle. \tag{138}$$

The commutator can be evaluated directly, noting the atomic potential term is $V = -Ze^2/(4\pi r) = -Z\alpha/r$:

$$\Big[\boldsymbol{p},[H,\boldsymbol{p}]\Big] = (-i)^2[\boldsymbol{\nabla},\,-\boldsymbol{\nabla}V] \tag{139}$$

$$= \nabla^2 V = 4\pi Z\alpha\,\delta^{(3)}(\boldsymbol{r}). \tag{140}$$

So, to evaluate the energy shift, we only need to evaluate the electron wavefunctions at the origin. From any textbook, we recall that only the s-states are non-zero at the origin, with

$$|\Psi_{ns}(0)|^2 = \frac{Z^3}{\pi a_0^3 n^3} = \frac{Z^3 m^3 \alpha^3}{\pi n^3} = \frac{2Z^3 m\alpha}{\pi n^3} R_y \tag{141}$$

and $|\Psi_{np}(0)|^2 = 0$ (finite nuclear size will modify this slightly, but will not massively change the result). Therefore, we have:

$$\delta\varepsilon_a^{\text{obs}} = \frac{8Z^4\alpha^3}{3\pi n^3} R_y \ln\left(\frac{\omega_{\text{cut}}}{\Delta_{\text{avg.}}}\right). \tag{142}$$

This splits the s and p levels, since the delta function in the matrix element means the p shift is zero. The shift also has the right sign, shifting the s state higher than the p state. Plugging in the numbers for hydrogrn, $Z = 1$, $n = 2$, and taking $\Delta_{\text{avg.}} = R_y$, $\omega_{\text{cut}} = m_e \simeq 0.5\,\text{MeV}$ (with $R_y \simeq 13\,eV$), we get

$$\delta E_{2s-2p_{1/2}} = \frac{\alpha^3}{3\pi}(13\,\text{eV}) \cdot 10 \approx 5 \times 10^{-6}\,\text{eV} \tag{143}$$

matching the observed shift!

Bethe's calculation of the Lamb shift demonstrated that QED was a useful theory for calculating accurate corrections, and demonstrated that renormalisation was possible. The observation of the Lamb shift won Lamb the 1955 Nobel Prize. Bethe was awarded a Nobel prize as well, in 1967, though for his work on astrophysics!

As a final remark – you may be left a little unsatisfied. We introduced an arbitrary cutoff into the calculation, leading to a linear divergence. After renormalisation (subtracting two linear divergences), we were left with a result that is logarithmically divergent. While this is much better than a linear divergence, it is still a divergence nonetheless. There is no solid justification for introducing the cut-off we did, and so your dissatisfaction is justified. The answer to this problem is that we have treated the problem non-relativistically. It is well beyond the scope of these notes, but it turns out that in a full relativistic treatment of the Lamb shift, the divergence is removed entirely. As it happens, in the relativistic case, the cut-off results directly in a logarithmic divergence, which is then removed entirely by the renormalisation process.

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