2 Classical Field Theory

This set of notes should be thought of as a *companion* to a good textbook, and is by no means complete. They are designed to give a broad overview of the most important topics in special relativity and classical field theory as briefly as possible. Familiarity with classical mechanics is assumed, as is an elementary understanding of vector calculus and electromagnetism. I recommend the books:

- L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (1971)
- D. J. Griffiths, *Intoduction to Electrodynamics* (1999)
- L. Susskind and A. Friedman, *Special Relativity and Classical Field Theory* (2014)
- J. D. Jackson, *Classical Electrodynamics* (2001)

2.1 The Principle of Relativity

In our study of classical mechanics, we saw that interactions between particles were described by means of potential functions, which depend on the relative positions of particles in the system, $V(\boldsymbol{x}_1 - \boldsymbol{x}_2)$. Such a description necessarily leads to the conclusion of the instantaneous propagation of interaction between particles: a change in the position of one particle immediately impacts the force on another, even if separated by very large distances. Such instantaneous interactions would appear to violate the concept of *locality* – that the motion of a particle should depend only on its immediate surrounds. More importantly, instantaneous interactions are shown by experimentation not to exist in nature. Therefore, we conclude that there must be a finite "speed of propagation" of interactions between bodies. In order to describe such propagation, as we shall see, we require a *field theory*, whereby the fields which mediate such interactions gain physical significance and should themselves follow equations of physical law. The study of field theory will result in a rich field of physics. In particular, it is required for a full (classical) description of electrodynamics, and will also form the basis from which we can study quantum field theory.

In order to quantitatively study any physical processes, we require a *reference system*, by which we mean a set of coordinates to keep track of the positions of

particles, as well as time intervals between events. We also remind the reader that there exist certain special reference frames, termed *inertial* reference frames, in which free particles do not accelerate. A pair of inertial reference frames move with respect to each other with constant velocity.

One of the key principles upon which classical mechanics was built was the *Galilean* principle of relativity, which encodes the concept of invariance of physical law between different inertial reference frames. One of the key properties of the Galilean relativity is that time is considered absolute; all observers in all reference frames share the identical time coordinate. At the same time, however, Maxwell's theory of electrodynamics implied that the speed of light (in vacuum), c, was a consequence of physical law, and, by the principle of relativity, should therefore be the same in all reference frames. The constancy of the speed of light has been verified by many experiments, including the famous experiment of Michelson and Morley. As we shall see, this observation is incompatible with the Galilean principle of relativity.

We shall build field theory from the Einstein *principle of relativity*, which shall supplant the Galilean principal of relativity:

1. The laws of physics are the same in all inertial reference frames,

2. The speed of light (in vacuum), c, is the same for all reference frames.

To see that these conditions are incompatible with the Galilean principle of relativity, consider two reference frames K and K', described by coordinates t and x, and t' and x', respectively, where the K' system moves relative to K (along the x axis) with speed v. The Galilean transformations linking the two reference frames are

$$x' = x - vt, \qquad t' = t, \tag{2.1}$$

where importantly, the time coordinate is considered universal and the same in all frames. Under this transformation, the speed of light is clearly not the same in both frames. If the path of a light ray in the K coordinates is $x = ct$, then in K' it would be $x' = ct - vt = (c - v)t$, so the ray moves with speed $c - v$.

Our first job, then, is to seek a set of transformations that will surpass the Galilean transforms, and preserve the constancy of c. It turns out that it is not possible to find such a transformation that leaves the concept of time universal between reference frames. An important consequence of this is that events which are simultaneous in one inertial reference frame may not be simultaneous in another.

2.1.1 The relativity of simultaneity

We shall begin by considering the canonical example. Consider a pair of observers, A and B, moving together along the x-axis at constant speed v relative to a stationary observer. We use coordinates t and x to label points as observed

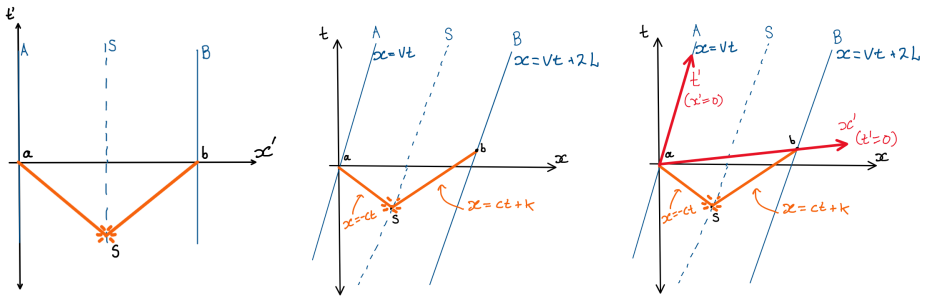
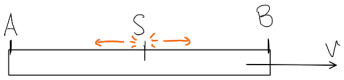


Figure 2.1: Space-time diagrams showing the relevant paths (*world lines*) in the co-moving K' frame, and the "stationary" K frame. The orange curves show the paths of the light rays, the blue lines show the paths of A and B, and the dotted line shows the path of the light source, S. The space-time point (*event*) of light emission is labelled s, and the events of the light reaching A and B are labelled a and b. The red lines show the K' axes ($x' = 0$ and $t' = 0$).

in the "stationary" K frame, and coordinates t' and x' in the K' frame that is co-moving with A and B. (Since there is no motion along the y or z axis, we need not consider them.) Equidistant between A and B is a light source, S, which moves along with them and emits a pulse of light:



We choose the origin of the coordinates in the each frame such that the light pulse reaches A at time $t = t' = 0$ and position $x = x' = 0$. The co-moving K' frame is termed the *rest frame* of observers A, S, and B, since in this frame they are at rest with respect to each other.

In the co-moving K' frame, the light pulse clearly reaches A and B at the same time ($t' = 0$), since and A and B are equidistant from the source. Therefore the *events* a and b (the space-time points where the signal reaches observer A and B) are *simultaneous*. However, in the stationary frame, it is seen that the light pulse reaches A *before* it reaches B. This is because the speed of light, c, is the same in all reference frames, but in the time it takes for the light pulse to reach the observers, A has moved towards the source, while B has moved away from it, so the light ray has to travel a longer distance to reach B. As a result, the events a and b are *not* simultaneous when observed from the stationary frame. This is shown in Fig. 2.1. It's possible to derive the set of coordinate transformations linking the reference frames geometrically, as is most common in textbooks. Here, we will take a slightly different route by first considering an invariant property called the *spacetime interval*.

2.1.2 The space-time interval

As demonstrated above, time intervals as measured in different references frames are not necessarily the same. By the assertion that the speed of light, c, is the same in all frames, this directly implies that lengths measured in different reference frames are also not invariant. This may be somewhat disturbing, since regular geometry is built on the concept of invariant lengths, and lengths are preserved under spatial rotations and the Galilean transformations. We are thus led to ask: is there an equivalent such invariant quantity within relativistic mechanics?

As is typical, we will begin our analysis by considering the paths taken by rays of light. Consider a light ray, omitted from point $\boldsymbol{x}_1$ at time t_1, and arriving at point $\boldsymbol{x}_2$ at time t_2. The distance the light ray moves is given by $|\Delta\boldsymbol{x}| \equiv |\boldsymbol{x}_2 - \boldsymbol{x}_1|$, or $\sqrt{\Delta\boldsymbol{x}^2}$.[1] At the same time, since the ray moves at the speed of light, the same distance can be expressed at $c\Delta t$. Therefore, we have the equality:

$$c^2\Delta t^2 - \Delta\boldsymbol{x}^2 = 0. \tag{2.2}$$

Clearly, since the speed of light is the same in all reference frames, the same equality holds in any reference frame:

$$c^2\Delta t'^2 - \Delta\boldsymbol{x}'^2 = 0. \tag{2.3}$$

Motivated by this equality, we define a quantity, *the interval* $\mathrm{d}s$ (or $\mathrm{d}\tau$)[2]:

$$\mathrm{d}s^2 \equiv c^2\mathrm{d}\tau^2 \equiv c^2\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2, \tag{2.4}$$

which is equal to zero for the interval between any two space-time points that can be connected by a light ray ("light-like" separated points), but in general will be non-zero.

As we saw, if the interval $\mathrm{d}s^2$ is zero in one frame, then it will be zero in every frame. It is natural to ask if this property holds for general intervals. To maintain the linearity, general intervals in different frames must be proportional:

$$\mathrm{d}s'^2 = f\,\mathrm{d}s^2, \tag{2.5}$$

where f is some (yet-undetermined) function. We note that due to the homogeneity of space, f cannot depend explicitly on time or coordinates. Further, due to the isotropy of space, it cannot depend on the *direction* of the velocity, but only its magnitude. Therefore, we must have $f = f(v^2)$.

[1] In our notation, bold symbols always represent regular three dimensional vectors, e.g., $\boldsymbol{x} = (x, y, z)$, and we use the shorthand $\boldsymbol{x}^2 \equiv x^2 + y^2 + z^2$.

[2] A common alternative definition is $\mathrm{d}s^2 = \mathrm{d}\boldsymbol{x}^2 - c^2\mathrm{d}t^2 = -c^2\mathrm{d}\tau^2$, as discussed below.

To proceed, consider a set of three reference frames, where frames 2 and 3 move relative to frame 1 with velocities $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$, respectively. We then have

$$\mathrm{d}s_2^2 = f(v_2^2)\,\mathrm{d}s_1^2, \qquad \mathrm{d}s_3^2 = f(v_3^2)\,\mathrm{d}s_1^2, \quad \text{and} \quad \mathrm{d}s_3^2 = f(v_{23}^2)\,\mathrm{d}s_2^2, \tag{2.6}$$

where $\boldsymbol{v}_{23}$ is the relative velocity of the third system with respect to the second (in Galilean relativity, this would be simply $\boldsymbol{v}_{23} = \boldsymbol{v}_3 - \boldsymbol{v}_2$, but in general is not). Solving for $f(v_{23}^2)$, we find:

$$f(v_{23}^2) = \frac{f(v_3^2)}{f(v_2^2)}. \tag{2.7}$$

Note, however, that $|v_{23}|$ depends not only on the magnitudes of the $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ vectors, but also on the angle between them. Since this angle does not appear on the right-hand-side of Eq. (2.7), the function $f(v^2)$ must be simply a constant, independent of the velocity.[3] Immediately, Eq. (2.7) also tells us this constant must be 1, which implies the interval $\mathrm{d}s$ is invariant.

A pair of space-time points (*events*) connected by interval $\mathrm{d}s^2 > 0$ are said to be *time-like* separated. Events with $\mathrm{d}s^2 < 0$ are said to be *space-like* separated. Since speeds may not exceed c, events which are space-like separated can have no causal contact, and can be considered *absolutely separated*. For any time-like separated points, there exists a frame of reference where the two events occur at the same position. Likewise, for space-like separated points, there exists a frame of reference where the two events happen at the same time; see Fig. 2.2.

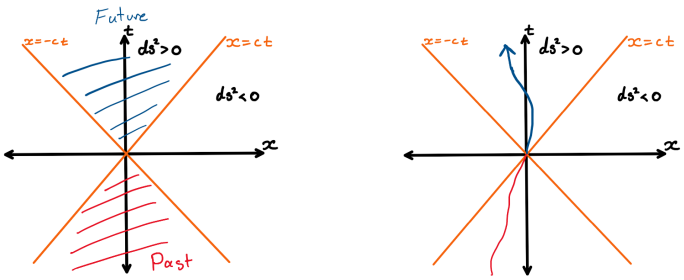


Figure 2.2: There is no frame where the events in upper region occur before the origin; they can therefore be considered strictly future points. The converse holds for the lower region. No points outside the *light cone* can reach (or be reached by) the origin; they are thus absolutely separated from the origin. The *world line* (space-time path) of any particle passing through the origin must remain inside this light cone.

[3] To see this more clearly, imagine frame 2 and 3 move with the same velocity with respect to frame 1 – clearly both sides of (2.7) are just 1, since $\boldsymbol{v}_{13} = \boldsymbol{v}_{12}$, and $\boldsymbol{v}_{23} = 0$. Now imagine changing the angle between the velocities of frames 2 and 3, keeping their magnitudes the same; $|v_{13}|$ and $|v_{12}|$ stay fixed by construction, but $|v_{23}|$ becomes non-zero.

2.2 Lorentz transformation

Transformations between inertial reference frames are termed *Lorentz transformations*. These include the familiar spatial rotations, as well as transformations between reference frames that move relative to each other with constant velocity, known as *Lorentz boosts* (or simply boosts). Since we understand that physics should remain invariant under global rotations of the spatial coordinates (the assumption of isotropy of space), we shall first consider boosts. Note that *translations* (in space and time) are not included in the Lorentz transformations; the generalised set of transformations which also includes translations is termed the Poincaré transformations.

We can use the invariance of the interval directly to work out the form of the Lorentz transformations. Consider two inertial reference systems. We label the coordinates of the first system t,x,y,z, and of the second t',x',y',z'. For simplicity, we will consider the case in which the second frame moves with constant speed v along the x-axis with respect to the first (due to the assumed invariance under spatial rotations, we are always free to choose any direction as the x-axis).

As we saw above, Galilean transformations, which leave the time coordinate invariant, cannot preserve the invariance of the interval. We therefore consider a general linear transformation between the two coordinates:

$$\begin{aligned} ct' &= \gamma ct + \eta x \\ x' &= \alpha x + \beta ct, \end{aligned} \tag{2.8}$$

where α, β, γ, η are as-yet undetermined dimensionless functions (as $v \to 0$, we have $\alpha, \gamma \to 1$, and $\beta, \eta \to 0$). The speed of light, c, is introduced to keep the dimensions of the equations the same. Since there is no motion along the y or z axes, we simply have $y' = y$ and $z' = z$. In order for the interval $\mathrm{d}s^2$ to be invariant, we require

$$\begin{aligned} c^2t^2 - x^2 &= c^2t'^2 - x'^2 \\ &= (\gamma^2 - \beta^2)c^2t^2 - (\alpha^2 - \eta^2)x^2 + 2(\gamma\eta - \alpha\beta)ctx. \end{aligned} \tag{2.9}$$

From this, we see that

$$\gamma^2 - \beta^2 = \alpha^2 - \eta^2 = 1, \quad \text{and} \quad \gamma\eta - \alpha\beta = 0, \tag{2.10}$$

which is three equations in four unknowns.

To break the degeneracy, we consider now the special case of the motion of the origin of the un-primed frame ($x = 0$). In the primed frame, this point moves with $x'/t' = -v$ by construction. By dividing the equations (2.8) and setting $x = 0$, we see that $\beta = -\gamma v/c$, which leaves three equations in three unknowns:

$$\begin{aligned} \gamma^2[1 - (v/c)^2] = \alpha^2 - \eta^2 &= 1 \\ \gamma(\eta + \alpha v/c) &= 0. \end{aligned} \tag{2.11}$$

From the third equation, we immediately see that $\eta = -v\alpha/c$, and from the first two, it is clear that $\alpha = \gamma$ and

$$\gamma(v) = \frac{1}{\sqrt{1-(v/c)^2}}. \tag{2.12}$$

(The sign ambiguity can be resolved by noting that $\gamma(0) = 1$.)

Putting things together, the Lorentz transformations for a boost along the x direction with speed v are

$$\begin{aligned} ct' &= (ct - \tfrac{v}{c}x)\gamma\,, \qquad & y' &= y\,, \\ x' &= (x - vt)\gamma\,, & z' &= z. \end{aligned} \tag{2.13}$$

The generalisation to boosts along an arbitrary axis is fairly clear (we are free to choose any direction as the x axis). There are several important observations to make. Firstly, the factor γ is only real valued for $|v| < c$, which implies the principle of relativity can only hold if nothing can move faster than the speed of light. Secondly, notice that by rearranging Eq. (2.13) and solving for x and t, we have

$$ct = (ct' + \tfrac{v}{c}x')\gamma, \quad \text{and} \quad x = (x' + vt')\gamma, \tag{2.14}$$

which is the same as Eq. (2.13) under $v \to -v$. The symmetry of this result is intuitive: we could have just as easily started from the primed system, with the unprimed system moving along the x' axis with the opposite speed $-v$.

Also, just as the notion of simultaneity was seen not to be consistent between reference frames, we find that both time intervals and lengths are not necessarily the same in different reference frames. This is known as time dilation and length contraction. For example, the length between A and the source S was observed to be L in the un-primed frame (see Fig. 2.1). It can be seen that this length is *shorter* than the length in the co-moving (or rest) frame, L_0, which is termed the *proper length*. In our case, we had $L = |x_S - x_A|$, while $L_0 = |x'_S - x'_A|$, leading to the relation for length contraction:

$$L = L_0/\gamma. \tag{2.15}$$

Using the same logic, and the assertion that the speed of light is the same in all frames, we also have

$$\Delta t = \Delta\tau\gamma, \tag{2.16}$$

where $\Delta\tau$ is the time interval measured in the rest frame. The quantity τ is known as the *proper time* – it is the time interval as measured in an observer's rest frame. With our convention [see Eq. (2.4)], proper time intervals are related to the space-time interval as $\mathrm{d}s = c\,\mathrm{d}\tau$.

We will now consider the important problem of the addition of velocities. Consider two frames, K and K', where K' moves with speed V along the x-axis

with respect to K. A particle moves with velocity v' in the K' frame. What is the velocity of the particle, v, as observed in the K frame? For the simplest case of all velocities co-linear along the x-axis, we have $v = dx/dt$, and $v' = dx'/dt'$. By diving the terms from Eq. (2.14), we immediately see:

$$v = \frac{v' + V}{1 + Vv'/c^2}. \tag{2.17}$$

As a final note, we mention that it is customary to work in so-called *natural units*, in which the speed of light takes the value

$$c = 1.$$

This can be done, for example, by measuring lengths in 'light seconds' (the distance covered by light in one second). This makes the equations much simpler, since we may drop the many factors of c that appear all over the place. In such units, velocities become dimensionless; as a result, the units for energies and mass become equivalent. We will use these natural units sometimes, though the c values will be kept in mot places. You should get used to swapping between regular and natural units.

***Problem* 2.1:** Repeat the derivation of Eq. (2.17) for the case of general velocity $\boldsymbol{v} = d\boldsymbol{x}/dt$, $\boldsymbol{v}' = d\boldsymbol{x}'/dt'$. You may still take the two frames to move along the x-axis with velocity V with respect to each other.

Answer (2.1): $v_x = \frac{v'_x + V}{1 + v'_x V/c^2}$, $v_y = \frac{v'_y \gamma(V)}{1 + v'_x V/c^2}$, $v_z = \frac{v'_z \gamma(V)}{1 + v'_x V/c^2}$.

2.2.1 Four vectors

We can arrange the three spatial coordinates and the time-coordinate into a kind of vector, known as a *four-vector* (sometimes called a Lorentz vector):

$$X = (ct, x, y, z)^{\mathrm{Tr}}. \tag{2.18}$$

The components of the four vector are typically denoted X^μ, with $X^0 = ct$, $X^1 = x$, etc. (the convention is that coordinate four vectors have length dimension). It is standard to use Greek indices to denote the components of four-vectors, running $0 - 3$, and reserve Latin indices for the spatial components $(1 - 3)$. As is standard, we reserve bold-type for regular three-dimensional vectors, and use regular type for four-vectors. This sometimes leads to confusion. As a result, it is common to use the index notation, and refer to x^μ as a four vector.[4] It is fairly common to see four-vectors written as $X = (t, \boldsymbol{x})$, or $x^\mu = (t, \boldsymbol{x})$.

With this concept, the Lorentz transformation can be written as a matrix equation,

$$X' = \Lambda X, \tag{2.19}$$

[4]Strictly speaking, of course, a^μ is the μ component of the vector a.

where (for a boost along the x-axis), we have:

$$\Lambda^{(x)}(v) = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.20}$$

Of course, we could have written this for a boost along a general direction, but the matrix becomes much more complicated. In general, we can always write $\Lambda = R^{-1}\Lambda^{(x)}R$, where R is the matrix for spatial rotations.

We will formalise this by taking *the definition* of a four vector to be an object that transforms between inertial reference frames according to Eq. (2.19), in much the same way that a regular three-dimensional vector is defined by its transformation under spatial rotations. In the same sense, we may define a *Lorentz scalar* (usually just called a scalar in this context), as being a quantity that is *invariant* under Lorentz transformations.

The concept of Lorentz scalars is very important. Since our goal is to construct a relativistic theory in which laws of physics are the same in all reference frames, it is clearly important to identify the invariant quantities (scalars) which are the same in all reference frames.

It's important to understand that four-vectors do not obey the rules of regular (Euclidean) geometry. For example, if we naively consider a "dot product"

$$X \cdot X = c^2t^2 + x^2 + y^2 + z^z \stackrel{?}{=} X' \cdot X'$$

we see from an application of the Lorentz transformation (2.13) that $X \cdot X \neq X' \cdot X'$, and so this quantity is *not* a scalar. Instead, it is the "Minkowski product" $c^2t^2 - \boldsymbol{x}^2$ that plays the role of the invariant, our four-dimensional space-time vectors form what is called "3 + 1-dimensional" *pseudo-Euclidean* space, or *Minkowski* space, which will explored in some more detail in the next section.

***Problem* 2.2:** Show explicitly that $\Lambda^{(x)}(v)^{-1} = \Lambda^{(x)}(-v)$. The physical meaning is clear: a boost of $+v$ followed by one of $-v$ must return us to the original frame.

2.2.2 Minkowski space

We can define a quantity, called the *metric tensor*:

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.21}$$

which allows us to simplify the expressions. For example, we can now write:

$$\mathrm{d}s^2 = \sum_{\mu\nu} \mathrm{d}x^\mu \eta_{\mu\nu} \mathrm{d}x^\nu. \tag{2.22}$$

We may now also *define* a new type of four-vector, denoted with a lowered index:

$$a_\mu \equiv \sum_\nu \eta_{\mu\nu} a^\nu. \tag{2.23}$$

We call the regular vectors *contravariant* (or "upstairs") vectors, while the other type are called *covariant* (or "downstairs") vectors. If the contravariant vectors have components $a^\mu \doteq (a_0, \boldsymbol{a})$, then the covariant vectors have $a_\mu \doteq (a_0, -\boldsymbol{a})$. Note that the covariant vectors transform as $a'_\mu = \Lambda_\mu{}^\nu a_\nu$, see Problem 2.3. We also have

$$a^\mu \equiv \sum_\nu \eta^{\mu\nu} a_\nu, \tag{2.24}$$

where $\eta^{\mu\nu} = \eta_{\mu\nu}$; a direct multiplication shows this is consistent with Eq. (2.23).

We will also introduce here the *Einstein summation convention*, where repeated indices are assumed to be summed over:

$$a^\mu b_\mu \equiv \sum_{\mu=0}^{3} a^\mu b_\mu. \tag{2.25}$$

Using this notation, the Lorentz transformation matrix Eq. (2.20) has the components $\Lambda^\mu{}_\nu$. It's important to keep track of the position of the indices when there are mixed upstairs and downstairs types, since, in general $T^\mu{}_\nu \neq T_\nu{}^\mu$. This notation leads to greatly simplified equations. For example, the interval formula (2.4) can be expressed compactly as

$$\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu, \tag{2.26}$$

and the Lorentz transform as

$$a'^\mu = \Lambda^\mu{}_\nu a^\nu. \tag{2.27}$$

It's easy to check explicitly that $a_\mu b^\mu = a^\mu b_\mu$.

It is possible to construct objects with more than one Lorentz index (we have already come across a few). For example, we can construct an object known as a *Lorentz tensor* (or simply a tensor) as $T^{\mu\nu} = A^\mu B^\nu$. It is clear that this tensor transforms according to

$$T'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}. \tag{2.28}$$

Certainly, not all tensors can be written as the product of two vectors, but we take the *definition* of a tensor to be an object which transforms according to Eq. (2.28). The number of Lorentz indices required to express the components of an object is referred to as its *rank*. In that sense, a scalar is a rank-0 tensor, a vector is a rank-1 tensor, the above tensor is rank 2, and so on. As a quick aside, we note that a common short-hand is often used for the "square" of four vectors and tensors: $a^2 = a_\mu a^\nu$, $T^2 = T_{\mu\nu} T^{\mu\nu}$.

It is appropriate here to note that our definition of the interval is not unique – we could just as easily have defined $\mathrm{d}s^2 = \mathrm{d}\boldsymbol{x}^2 - \mathrm{d}t^2$. In fact, both choices are commonly made. The other choice corresponds to taking the opposite sign for the metric η (2.21). This is usually denoted as the $(+---)$ metric (with the *metric signature* of -2) for our choice, or the $(-+++)$ metric (with the metric signature $+2$) for the other. The $+2$ metric is most common among those studying general relativity, while the -2 metric is most common in relativistic quantum mechanics. One must take particular care when mixing formulas taken from different works.

***Problem* 2.3:** Derive the transformation law for covariant vectors, and show they transform like contravariant ones under $v \to -v$ (i.e., by inverse transform, see Problem 2.2).
Solution (2.3): For the contravariant vectors, we have $A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu}$, and the covariant vector is defined $A'_{\mu} \equiv \eta_{\mu\nu} A'^{\nu}$. Therefore,

$$A'_{\mu} = \eta_{\mu\rho}\Lambda^{\rho}{}_{\nu}A^{\nu} = \eta_{\mu\rho}\Lambda^{\rho}{}_{\sigma}\eta^{\sigma\nu}A_{\nu} = \Lambda_{\mu}{}^{\nu}A_{\nu}.$$

A straight-forward matrix multiplication $\eta\Lambda\eta$ shows the components of $\Lambda_{\mu}{}^{\nu}A_{\nu}$ are like those of $\Lambda^{\mu}{}_{\nu}A^{\nu}$ with $v \to -v$. (Be careful with the notation: $M_{\mu\nu}v_{\nu} \neq M_{\nu\mu}v_{\nu}$.)

***Problem* 2.4:** Prove explicitly that $a_{\mu}b^{\mu}$ is a scalar (i.e., the same in all frames). It is enough to consider boosts along a single axis, noting that Euclidean distances are invariant under rotations.
Answer (2.4): Follows directly from Problem (2.3).

***Problem* 2.5:** Show explicitly that the metric tensor is the same in all frames.

2.2.3 Derivatives, four-velocity

In regular three dimensional space, we can form vectors from the derivative of scalar functions, $\boldsymbol{\nabla}\phi$. Can we likewise form four-vectors from four-derivatives? In other words, is the quantity

$$\frac{\partial\phi}{\partial x^{\mu}} = \left(\tfrac{\partial\phi}{\partial x^0}, \tfrac{\partial\phi}{\partial x^1}, \tfrac{\partial\phi}{\partial x^2}, \tfrac{\partial\phi}{\partial x^3}\right) \tag{2.29}$$

a four vector (does it transform between frames according to Lorentz transformations)? It's easy to check, since we know how $\mathrm{d}x$ transforms. Considering again the case of boosts along the x-axis, from Eq. (2.14) we have

$$\frac{\partial t}{\partial t'} = \frac{\partial x}{\partial x'} = \gamma \quad \text{and} \quad \frac{c\,\partial t}{\partial x'} = \frac{\partial x}{c\,\partial t'} = \frac{v}{c}\gamma.$$

Therefore,

$$\begin{aligned} \frac{\partial\phi}{c\,\partial t'} &= \frac{\partial\phi}{c\,\partial t}\frac{\partial t}{\partial t'} + \frac{\partial\phi}{\partial x}\frac{\partial x}{c\,\partial t'} = \left(\frac{\partial\phi}{c\,\partial t} + \frac{v}{c}\frac{\partial\phi}{\partial x}\right)\gamma, \qquad \text{and} \\ \frac{\partial\phi}{\partial x'} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial\phi}{\partial t}\frac{\partial t}{\partial x'} = \left(\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{c\,\partial t}\right)\gamma. \end{aligned} \tag{2.30}$$

Comparing with Eqs. (2.13) or (2.20), we see that this derivative *nearly* transforms as a four vector – it transforms as though through a Lorentz transformation with the opposite sign for the velocity. That is, it transforms as a *covariant* vector (see Problem 2.3). We introduce the standard notation for the *covariant derivative* operator,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\nabla}, \tag{2.31}$$

which makes the covariance of the derivative explicit. It is also useful to define the *contravariant derivative*, $\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$. With this, we can write simply the d'Alembertian operator $\partial^2 \equiv \partial_\mu\partial^\mu = \frac{1}{c^2}\partial_t^2 - \nabla^2$ (often denoted $\Box$).

Now that we have a solid definition for derivatives, we may be tempted to form a velocity vector as $v^\mu = \mathrm{d}x^\mu/\mathrm{d}t$. It is not difficult to see, however, that this quantity is not a valid four vector (while x^μ is a Lorentz vector, t is not a Lorentz scalar). Instead, we can take the derivative of position with respect to *proper time*, τ

$$u^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \tag{2.32}$$

which we call the *four velocity*. Since τ is a scalar, u is a vector.

We can see the connection to the regular velocity, $\boldsymbol{v}$, by examining the components:

$$u^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}. \tag{2.33}$$

Since $\mathrm{d}\tau = \sqrt{\mathrm{d}s^2/c^2} = \mathrm{d}t\sqrt{1 - \mathrm{d}\boldsymbol{x}^2/(c\,\mathrm{d}t)^2} = \mathrm{d}t\sqrt{1-(v/c)^2}$, we see the temporal and spatial components of U are given by

$$u^0 = c\,\gamma, \quad \text{and} \quad u^i = v^i\gamma. \tag{2.34}$$

Note that, by its definition,

$$u^\mu u_\mu = c^2. \tag{2.35}$$

As such, the four velocity only has three independent components.

***Problem* 2.6:** Show that $\mathrm{d}t\,\mathrm{d}V$ is a scalar (hint: you may use the invariance of volumes $\mathrm{d}V$ under rotations to arbitrarily choose the x-axis).

2.3 Relativistic dynamics

We wish to find the rules that govern the dynamics of particles within the framework of the principle of relativity. Following the same logic from classical mechanics, the equations of motion should follow from the *principle of stationary action* (see Mechanics Sec. 1.2). For the equations of motion to be the same in all inertial reference frames, we need an action that is the same in all inertial

reference frames – i.e., the action should be a Lorentz scalar. The simplest scalar is $\mathrm{d}s$, so we can try an action of the form

$$S = kc \int \mathrm{d}s, \tag{2.36}$$

where k is an arbitrary constant, and c is introduced for dimensional convenience. To be dimensionally correct, the k constant should have mass dimensions.

In Mechanics Sec. 1.3, we found based on symmetry arguments that the Lagrangian for a free particle should be a function of v^2 only. Those arguments should hold in the relativistic case, and indeed they do. To put things into more familiar form, note that $\mathrm{d}s = \sqrt{c^2\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2}$. By pulling out a factor of $\mathrm{d}t$, we may write

$$S = kc^2 \int \sqrt{1 - (v/c)^2}\, \mathrm{d}t. \tag{2.37}$$

Now, the integrand may be recognised as a Lagrangian $L = kc^2/\gamma$. Consider the non-relativistic expansion, that is, the expansion around small v:

$$L \approx kc^2 - k\frac{v^2}{2} - \frac{kv^4}{8c^2} - \dots \tag{2.38}$$

The first term is simply a constant, which will not affect the equations of motion. Therefore, it is the second term that is important in the non-relativistic limit. Note that if we set $k = -m$, then the action recovers the non-relativistic equations of motion (for a free particle) [Mechanics Eq. (1.16)]. Therefore, we have the relativistic Lagrangian for a free particle:

$$L = -mc\frac{\mathrm{d}s}{\mathrm{d}t} = -mc^2/\gamma. \tag{2.39}$$

The canonical momentum, p_i (momentum conjugate to general coordinate q_i), can be determined from the Lagrangian [Mechanics (1.18)]

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{2.40}$$

from which we find the relativistic expression for the linear momentum:

$$\boldsymbol{p} = m\boldsymbol{v}\gamma. \tag{2.41}$$

Likewise, the energy (Hamiltonian) of the free particle can be determined from

$$H = p_i\dot{q}_i - L \tag{2.42}$$

[see Mechanics (1.35)], from which we find:

$$\mathcal{E} = mc^2\gamma. \tag{2.43}$$

Expanding around small v,

$$\mathcal{E} \approx mc^2 + \frac{mv^2}{2} + \frac{3mv^4}{8c^2} + \dots \tag{2.44}$$

Therefore, in the low-velocity limit, the energy does not tend to zero, but to a constant – the famous value mc^2. The term $\propto v^4$ can be considered the lowest-order relativistic correction to the kinetic energy.[5]

For a particle in a potential, we have for the Lagrangian

$$L = -mc^2/\gamma - \phi(\boldsymbol{x}). \tag{2.45}$$

A straight-forward application of the the Euler-Lagrange equations leads to

$$m\frac{\mathrm{d}(\boldsymbol{v}\gamma)}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\nabla\phi, \tag{2.46}$$

which is the relativistic version of Newton's second law. We therefore define the *force* vector

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}, \tag{2.47}$$

which may be written as $\boldsymbol{F}\gamma = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}$.

2.3.1 Four-momentum

Above, we found relativistic expressions for the linear momentum and energy of a particle. It's clear that the energy of a particle is *not* a Lorentz scalar – it depends on the relative velocity of the observer with respect to the particle. At the same time, the relativistic momentum was a three-vector, not a Lorentz four-vector. If we require sets of relativistically invariant equations, we must seek a relativistically invariant way of expressing these important quantities. In other words, is there a way to form a four vector from these quantities?

A natural stating point is to consider the four-velocity, u^μ. We define a quantity p^μ by multiplying u^μ by mass

$$p^\mu = mu^\mu. \tag{2.48}$$

Since the four velocity is a Lorentz vector so is p^μ. From Eq. (2.34), we see immediately that the spatial components of p^μ are exactly the relativistic momentum. Likewise, from Eqs. (2.34) and (2.43), the temporal component is the relativistic energy (divided by c). We call this vector the *four momentum*, which has components $p^\mu = (\mathcal{E}/c, \boldsymbol{p})$.

[5] In old works, the term "rest mass" may be defined via $m = m_0\gamma$. This terminology is no longer used; mass m is understood to be a scalar parameter of the theory. We may instead define "rest energy" as $\mathcal{E}_0 = mc^2$.

Just as the four velocity has only three independent components (2.35), the same is true for the four momentum. The components of the four momentum are restricted by the condition

$$p^\mu p_\mu = \mathcal{E}^2/c^2 - \boldsymbol{p}^2 = m^2c^2, \tag{2.49}$$

which follows directly from the above expressions. Rearranging, we come to the famous Einstein energy-momentum relation:

$$\mathcal{E}^2 = m^2c^4 + \boldsymbol{p}^2c^2, \tag{2.50}$$

which (of course) is the same as Eq. (2.43).

Note that the above Lagrangian formalism clearly does not work in the case of massless particles. This is a result of the fact that there exists no reference frame where massless particles are at rest. While the relations in Eq. (2.43) and (2.41) are not valid for massless particles, Eq. (2.50) *is* valid. For such particles, this relation implies $|p| = \mathcal{E}/c$.

2.4 Classical fields

In general, field theory is the study of continuous distributions, which can be described by continuous functions of time and coordinates: $\phi = \phi(t, \boldsymbol{x})$. While classical fields may describe (near-)continuous distributions of matter, we will focus on the case of the intrinsically continuous fields that mediate the interaction between bodies (e.g., electromagnetic or gravitational fields). Our goal, in general, is to ascertain the set of *field equations* which govern the temporal and spatial evolution of the fields.

Without any assertion for the specific form these equations may take, we can follow the exact same logic as in classical mechanics, and assert they obey a principle of stationary action, with action S defined:

$$S = \int \mathrm{d}t\, L, \tag{2.51}$$

where the Lagrangian L depends on the fields $\phi(t, \boldsymbol{x})$ and their derivatives, $\partial_\mu\phi$. From the principles of homogeneity and isotropy, it cannot depend explicitly on coordinates [e.g., see Mechanics Sec. 1.3].

Further, if we require relativistically invariant equations of motion, then the action should also be relativistically invariant, i.e., a Lorentz scalar. Note, however, that if S is a scalar, then the Lagrangian L in Eq. (2.51) certainly cannot be, since $\mathrm{d}t$ is not a scalar. For this reason, we introduce the *Lagrangian density*[6],

[6]It is common to simply use the term Lagrangian to refer to the Lagrangian density in the context of field theories.

$\mathcal{L}$, defined via $L = \int \mathrm{d}^3x\, \mathcal{L}$, such that

$$S = \int \mathrm{d}t\, \mathrm{d}V\, \mathcal{L} = \frac{1}{c} \int \mathrm{d}^4x\, \mathcal{L}, \tag{2.52}$$

where and $\mathrm{d}^4x \equiv c\, \mathrm{d}t\, \mathrm{d}x\mathrm{d}y\mathrm{d}z$. Since $\mathrm{d}t\, \mathrm{d}V$ *is* a scalar (see Problem 2.6), so is $\mathcal{L}$.

The derivation of the Euler-Lagrange equations for a field theory follows closely that in the non-relativistic case; we seek an extremum of the action such that $\delta S = 0$. Since the Lagrangian depends on the field ϕ and its derivatives $\partial_\mu \phi$, the general variation of the action is

$$\begin{aligned} \delta S &= \frac{1}{c} \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right] \\ &= \frac{1}{c} \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi, \end{aligned} \tag{2.53}$$

where in the first line we used the commutativity of the derivative to write $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$, and in the second line we used integration by parts (assuming the boundary term goes to zero). For δS to be zero for general variations $\delta\phi$, the term in the square brackets must be zero, yielding the Euler-Lagrange equations for the field ϕ

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right). \tag{2.54}$$

The generalisation to the case of multiple fields $(\phi_1, \phi_2, \ldots)$ is clear; just as in the case of multiple particles in classical mechanics, there is a separate Euler-Lagrange equation for each field. While we explicitly wrote the equations as though ϕ were a scalar field, the derivation is exactly the same for vector (and higher rank) fields $\phi \to \phi^\mu$: there is a Euler-Lagrange equation for each field component. It's important to realise that any term added to the Lagrangian density that is a total divergence

$$\mathcal{L} \to \mathcal{L} + \partial_\mu k^\mu, \tag{2.55}$$

will change the action by at most a boundary term (due to the divergence theorem). If the source k^μ goes to zero at infinity, which is usually the case, then such a term cannot impact the field equations.

Notice what happens in the special case that the field has *no* spatial degrees of freedom, i.e., $\phi = \phi(t)$. The spatial derivatives are zero, and the Euler-Lagrange equations become exactly those from regular (particle) classical mechanics, with $\phi(t)$ playing the role of $x(t)$. A field theory with no spatial degrees of freedom reduces to regular particle mechanics.

2.4.1 Field theory as the continuum limit

As a first example of a field theory, we'll consider a simple non-relativistic one dimensional system of coupled harmonic oscillators, which you may have seen before. Take a large set of particles, each of mass m, connected by a series of springs of spring constant k. In equilibrium, the masses are separated by Δx.

To write the Lagrangian, note that the kinetic energy of each particle is $m\dot{\phi}_n^2/2$, where ϕ_n denotes the deviation of the nth particle from its equilibrium position. There is also a potential energy associated with each spring (i.e., with each pair of particles) given by Hooke's law $k(\Delta\phi_n)^2/2$, where $\Delta\phi_n = \phi_n - \phi_{n-1}$. Finally, we might consider the case where each mass is also held in position by a separate spring with constant k'. Then, the Lagrangian is

$$L = \sum_n \left(\frac{m}{2}\dot{\phi}_n^2 - \frac{k}{2}(\Delta\phi_n)^2 - \frac{k'}{2}\phi_n^2 \right). \tag{2.56}$$

To see how this problem can be treated using field theory, we shall take the continuum limit, where the separation between each mass $\Delta x \to 0$. To aid in taking the limit, we factor out Δx

$$L = \Delta x \sum_n \left(\frac{m}{2\Delta x}\dot{\phi}_n^2 - \frac{k\Delta x}{2}\frac{(\Delta\phi_n)^2}{(\Delta x)^2} - \frac{k'}{2\Delta x}\phi_n^2 \right). \tag{2.57}$$

Note that we could write instead $\phi_n = \phi(x_n)$, where x_n is location of the nth particle. It's important to note that it is ϕ, not x_n, that is the *dynamical variable*; x_n is simply the label for which mass is being referred to.

In taking the $\Delta x \to 0$ limit, we recognise $\Delta\phi_n/\Delta x$ as the derivative $\partial_x\phi$. We further define $\mu = m/\Delta x$ (mass density), $\tau = k\Delta x$ (tension), and $\sigma = k'/\Delta x$ (stiffness), leading to

$$L = \int \mathrm{d}x \left[\frac{\mu}{2}(\partial_t\phi)^2 - \frac{\tau}{2}(\partial_x\phi)^2 - \frac{\sigma}{2}\phi^2 \right]. \tag{2.58}$$

The equations of motion are

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\tau}{\mu}\frac{\partial^2\phi}{\partial x^2} + \frac{\sigma}{\mu}\phi = 0. \tag{2.59}$$

In the simplest case of $\sigma = 0$, the solutions are just plane waves travelling with speed $v = \sqrt{\tau/\mu}$. In the case where $\tau = 0$, the equations are just those of the simple harmonic oscillator with $\omega = \sqrt{\sigma/\mu} = \sqrt{k'/m}$, which is of course not surprising. The generalisation to three spatial degrees of freedom is clear: $\partial_x \to \nabla$.

2.4.2 Relativistic field theory

In general, we can consider Lagrangian's for relativistic field theories that satisfy the following principles:

- The Lagrangian density should be a *local* function of the fields and their derivatives (i.e., involve only a single coordinate)
- It should depend on terms that are at most first-order time derivatives of the fields, which ensures the equations of motion involve at most second-order time derivatives
- It should be a Lorentz scalar to ensure relativistic invariance (which, together with the previous point, implies also at most first-order spatial derivatives).

To form a Lorentz invariant Lagrangian density from the fields, we must understand what happens to the fields under Lorentz transformations. In the simplest case of a scalar field, the meaning is reasonably intuitive. A scalar field is simply a function that assigns a number – a scalar – to every space-time location. By definition, scalars remain invariant under Lorentz transformations, so we have

$$\phi(x^\mu) \to \phi'(x^{\mu\prime}) = \phi'(\Lambda^\mu{}_\nu x^\nu) = \phi(x^\mu), \tag{2.60}$$

where ϕ' is the field as written in the transformed coordinates, which we could write as $\phi'(y) = \phi(\Lambda^{-1}y)$, where $y = x' = \Lambda x$. That is, the value of the transformed field at the transformed coordinate should be the same as the original field at the original coordinate.

In much the same way, a vector field is a function that assigns a *vector* to every space-time location. By definition, vectors transform according to Eq. (2.27), so we have

$$A^\mu(x) \to A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x). \tag{2.61}$$

The extension to tensor and higher-rank forms carries in the same way.

There is another constraint we may place on the Lagrangian that holds in many (but certainly not all) cases of physical interest. Many fields in nature, for example electromagnetic and gravitational fields, obey the *principle of superposition*. If there are two fields, ϕ_1 and ϕ_2, which independently satisfy the field equations and obey the superposition principle, then total field is the simple composition of each, $\Phi = \phi_1 + \phi_2$, which must also obey the field equations. In other words, if if ϕ_1 and ϕ_2 are both solutions to the field equations, then so must be $\phi_1 + \phi_2$. The field equations of motions resulting from the Euler-Lagrange condition (2.54) are differential equations. For the superposition principle to hold, these must be *linear* differential equations. That is, the field equations of motion must contain terms involving only linear functions in ϕ. In varying the

action (i.e., applying the Euler-Lagrange equations), the degree of the terms in the Lagrangian is reduced by one. This implies the further condition:

- If the field obeys the superposition principle, the Lagrangian density may contain terms that are at most quadratic in the field.

This is not a universal constraint; in general, fields need not obey such a superposition principle, and the Lagrangian can have any power. However, it's important to note that higher powers may be important at certain energy scales only.

The simplest Lorentz scalar that is quadratic in the fields and containing only first-order derivatives is $\partial_\mu\phi\partial^\mu\phi$. Note this is exactly the first two terms from our simple example in Eq. (2.58) in the case $\tau = \mu$. We may then also consider possible potential terms of higher field powers:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - c_2\phi^2 - c_3\phi^3 - c_4\phi^4 - \dots, \tag{2.62}$$

An important physical consequence for cases such as these is that the behaviour of the field may change significantly with the amplitude of the field. For example, for small field perturbations, the quadratic term will dominate. For $\phi \gtrsim c_2/c_3$, the cubic term will become important, and so on. This is the general principle underlying *effective field theories*.

2.4.3 Scalar field theory: Klein-Gordon equation

We shall now briefly consider an important example, the classical Klein-Gordon equation for a real scalar field ϕ. The action for a scalar field may be written

$$S = \int \mathcal{L}(\phi, \partial_\mu\phi)\,\mathrm{d}^4x. \tag{2.63}$$

(We use $c = 1$ units for the remainder of this section.) As we discussed above, one of the simplest examples for an invariant Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\mu^2}{2}\phi^2, \tag{2.64}$$

where the signs and factors are chosen by convention. (It should be noted that while the overall sign is arbitrary, the relative sign is of physical significance.) Note that we cannot include terms of the form $\partial^2\phi$, since we require only first-order derivatives, and we restrict ourselves to quadratic terms in ϕ in order to preserve the superposition principle (i.e., so that we have linear equations of motion). Since the Lagrangian density has dimension energy per volume, $[\mathcal{L}] = E/L^3$, the field ϕ has dimension: $[\phi] = E^{1/2}\,L^{-1/2}$.

It is straight forward to find the field equations. From Eq. (2.54), we have

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\mu^2\phi, \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi. \tag{2.65}$$

From the Euler-Lagrange condition (2.54), this leads to the equations of motion for the Klein-Gordon field:

$$\partial^2\phi + \mu^2\phi = 0, \tag{2.66}$$

which you may recognise as a wave(-like) equation. It is not difficult to see that solutions include those of the form of (real) combinations of oscillating plane waves:

$$\phi = e^{\pm ik_\mu x^\mu}. \tag{2.67}$$

These describe classical waves with angular frequency $\omega = ck_0$, and wave vector $\boldsymbol{k}$. On inserting Eq. (2.67) into (2.66), we find the condition for the Klein-Gordon wave four vector:

$$k_\mu k^\mu = \mu^2. \tag{2.68}$$

2.4.4 Particles and fields

In a quantum field theory, particles and fields will become unified, and will be described by a single quantum Lagrangian. Classically, however, they are treated differently. If we consider a set of particles, described by coordinates $\boldsymbol{x}$, as well as a set of fields ϕ, then the total Lagrangian may be expressed as the sum of terms involving only particle coordinates, terms involving only fields, and interaction terms, which involve both:

$$L = L_{\text{particle}}(\boldsymbol{x}) + L_{\text{field}}(\phi) + L_{\text{interaction}}(\boldsymbol{x}, \phi) \tag{2.69}$$

(for simplicity, we write the equations for a single particle and a single field, though the generalisation is clear). It is no surprise that if there is no interaction between the particles and the field, then they evolve independently from one another, and may be considered separately. If the interaction term is non-zero, however, the equations of motion will not be separable; physically this means that the field impacts how the particle evolves, and likewise, the particle impacts how the field evolves.

In certain circumstances, a simplifying approximation is possible. If the interaction between the particle and the field is sufficiently weak, then the action of the particle on the field may be neglected (consider, e.g., small mass in a gravitational field, or an electron in a macroscopic electric field). In that case, supposing the values for the fields are known, we need only consider the particle and interaction terms in the problem.

Let's consider an example of the interaction between a particle scalar field. Since the action must be a scalar, the simplest action we can write down involving both particle and field terms is

$$S_{\text{int}} = -\int f(\phi)\,\mathrm{d}s, \tag{2.70}$$

where $f(\phi)$ is any scalar function of ϕ. Note that the values of $\phi(x^\mu)$ are evaluated at the positions (along the world line) of the particle, and the arbitrary negative sign is by convention. In analogy with the pure particle case (2.39), the total Lagrangian is then (with $c = 1$)

$$L_{\text{part+int}} = -\left[m + f(\phi)\right]\sqrt{1 - v^2}. \tag{2.71}$$

By comparison with Eq. (2.39), it's clear the effect of this term on the equation of motion of the particle is $m \to m + f(\phi)$. This is a (very crude) analogy for how the Higg's field (which is a scalar field, albeit a quantum one) may give mass to particles.

2.5 Symmetries and conservation laws

Following our discussion of symmetries in classical mechanics [Sec. 1.4], we shall consider the extension to relativistic field theories. Consider a general continuous transformation of the form $\phi \to \phi + \delta\phi$. We consider infinitesimal transformations, such that

$$\delta\phi = \epsilon f, \tag{2.72}$$

where ϵ is an infinitesimal continuous parameter, and finite continuous transformations can always be built from successive applications of the above. While we explicitly write the equations as though for scalar fields, the arguments hold for the general case. In this section, we use $c = 1$.

This transformation is considered a *symmetry* if it leaves the equations of motion unchanged. From the principle of stationary action, this means it may change the action by at most a constant. In other words, the change in the Lagrangian $\delta\mathcal{L}$ must be either zero, or a total divergence. Therefore, in the assumption that Eq. (2.72) is a symmetry, we have

$$\delta\mathcal{L} = \epsilon\partial_\mu k^\mu, \tag{2.73}$$

for some k^μ.

The general variation in the Lagrangian from the variation $\delta\phi$ is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi). \tag{2.74}$$

From the equations of motion (2.54), we have

$$\begin{aligned}\delta\mathcal{L} &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \\ &= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right].\end{aligned} \tag{2.75}$$

Comparing to Eq. (2.73), we thus have

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} f - k^\mu \right] = 0, \tag{2.76}$$

where the term in the square brackets defines a conserved *current density*, J^μ:

$$\partial_\mu J^\mu = 0, \quad \text{with} \quad J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} f - k^\mu. \tag{2.77}$$

This important result is called *Noether's theorem*[7]:

> *For every continuous symmetry of the action, there a corresponding locally-conserved charge and current.*

2.5.1 Continuity equations

Note that Noether's theorem (2.77) is a similar, but a much stronger, result than we found in the classical case [see Mechanics Sec. 1.4.4], where we had

$$\frac{\mathrm{d}}{\mathrm{d}t} Q = 0, \quad \text{with} \quad Q \equiv Pf. \tag{2.78}$$

By comparison, we could define the conserved *charge*,[8] as the volume integral of the zero component of the conserved current density:

$$Q = \int \mathrm{d}^3 x \, J_0.$$

The classical case, Eq. (2.78), implies a globally conserved charge, Q. The relativistic case, Eq. (2.77), implies the stronger condition of a continuously conserved current:

$$\frac{\partial}{\partial t} J_0 + \nabla \cdot \boldsymbol{J} = 0, \tag{2.79}$$

which is a *continuity equation*. This implies a *locally* conserved four-current, rather than simply globally conserved charge. To see this, integrate over any arbitrary volume, V:

$$\int \mathrm{d}V \left(\frac{\partial}{\partial t} J_0 + \nabla \cdot \boldsymbol{J} \right) = \frac{\partial}{\partial t} Q + \int \mathrm{d}V \, (\nabla \cdot \boldsymbol{J}) = 0 \tag{2.80}$$

$$\implies \quad \frac{\partial}{\partial t} Q = - \oint_S \boldsymbol{J} \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{2.81}$$

[7] E. Noether, "Invariante Variationsprobleme", Gott. Nachr. **1918**, 235–257 (1918), *Invariant Variation Problems (Translation)*, arXiv:physics/0503066.

[8] We use the word "charge" here generally to refer to any conserved quantity.

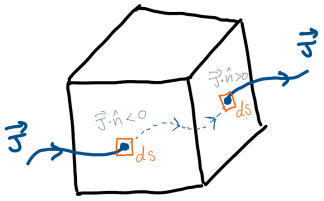


Figure 2.3: The flux of current $\boldsymbol{J}$ through surface, S, of a volume, V, is the integral of $\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S$ over the surface, where $\hat{\boldsymbol{n}}$ in the unit vector normal to the surface element $\mathrm{d}S$. By Eq. (2.81), this is equal to the (negative) of the rate of change of the total charge $Q=\int J^0\mathrm{d}V$ in the volume.

where we used Gauss' theorem in the final step [see Eq. (2.167) in Appendix 2.9.2], S is the surface of the volume V, and $\boldsymbol{n}$ is the unit vector perpendicular to the surface. This says that the change in the charge inside any arbitrary volume V [left-hand-side of (2.81)], must be compensated for by a corresponding flux of current through the surface of the volume [right-hand-side of (2.81)]; see Fig. 2.3.

For the particular case of *coordinate transformations*, $x^\mu\to x^\mu+\delta x^\mu$ (e.g., translations, rotations, boosts), we have

$$\delta\phi=(\partial_\mu\phi)\,\delta x^\mu,\quad\text{and}\quad\delta\mathcal{L}=(\partial_\mu\mathcal{L})\delta x^\mu. \tag{2.82}$$

In many cases of interest (including translations and rotations), the divergence of the coordinate transform is zero: $\partial_\mu\delta x^\mu=0$. Then, we may write the variation in the Lagrangian particularly simply

$$\delta\mathcal{L}=\partial_\mu(\mathcal{L}\delta x^\mu), \tag{2.83}$$

which allows us to write the conserved current as

$$J^\mu=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\epsilon}-\frac{\delta x^\mu}{\epsilon}\mathcal{L}. \tag{2.84}$$

2.5.2 Stress-energy tensor

In Mechanics, we saw that the symmetry of a system under temporal translations led to the law of conservation of energy, and that the symmetry under spatial translations led to momentum conservation. We can now extend this idea to the relativistic scenario. Consider space-time translations of the form

$$x^\nu\to x^\nu+\delta x^\nu, \tag{2.85}$$

where δx^ν is a constant. At first, to get used to manipulating multi-index equations, you may consider ν to represent a single fixed (yet arbitrary) coordinate

index. Under this translation, we have again

$$\delta\phi = \partial_\mu\phi\,\delta x^\mu$$

where, in comparison with the above, we set $f = \partial_\nu\phi$, and $\epsilon = \delta x^\nu$.

Therefore, using Noether's theorem, we have the conserved current:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \partial_\nu x^\mu\mathcal{L}. \tag{2.86}$$

where $\partial_\nu x^\mu = \delta^\mu_\nu$ is 1 for $\mu = \nu$, and zero otherwise. We will have one such equation for each ν index. The resulting set of equations define the components of a rank two *tensor*, which we term the *stress energy tensor*:

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}, \tag{2.87}$$

where we used Eq. (2.24) to raise the second index. Note that by construction (2.76), we have

$$\partial_\mu T^{\mu\nu} = 0, \tag{2.88}$$

so long as space-time translation is a symmetry of the system.

2.5.3 Hamiltonian and momentum

As we saw in our study of classical mechanics, symmetry under time translations (i.e., the above with $\nu = 0$), led to energy conservation. In this case, considering the $\nu = 0$ terms, we have

$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} + \partial_i T^{i0} \tag{2.89}$$

$$= \partial_t T^{00} + \partial_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\dot{\phi}\right) = 0, \tag{2.90}$$

which can be understood as a continuity equation for energy density. We can thus recognise T^{00} as the *energy density*, or *Hamiltonian density*:

$$T^{00} \equiv \mathcal{H} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi} - \mathcal{L}. \tag{2.91}$$

Note the similarity to the classical expression for the Hamiltonian (2.42). The continuity equation is therefore

$$\frac{\partial}{\partial t}\mathcal{H} = -\nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\dot{\phi}\right), \tag{2.92}$$

and we can further recognise T^{i0} (the term in the parenthesis) as the *energy density flux*.

Similarly, T^{0i} can be recognised as the momentum density

$$P^i \equiv \int \mathrm{d}^3x \, \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\boldsymbol{x})} \partial^i \phi. \tag{2.93}$$

In analogy with the classical expression (2.40), we also define the *canonical momentum density* for the field

$$\pi(\boldsymbol{x}) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\boldsymbol{x})}. \tag{2.94}$$

With this, we may write the expression for the momentum as

$$\boldsymbol{P} = -\int \mathrm{d}^3x \, \pi(\boldsymbol{x}) \, \boldsymbol{\nabla} \phi, \tag{2.95}$$

which is the quantity that is conserved if the system is invariant under spatial translations. This can be written in the four-vector form as $\Pi^\mu = T^{0\mu}$, or

$$P^\mu = \int \mathrm{d}^3x \, T^{0\mu}. \tag{2.96}$$

These definitions may seems a little abstract for now. One way to justify the identification of these terms as the energy and momentum is to consider the limit of a field with zero spatial degrees of freedom: $\phi(t, \boldsymbol{x}) \to \phi(t)$. As we discussed earlier, in this limit, we recover regular particle mechanics when identifying $\phi(t) = x(t)$. It's straight forward to show that the above expressions for T^{00} and T^{0i} do reduce to the classical definitions of energy and momentum (density) in this limit. The physical meaning of these terms will become much clearer once we consider specific examples.

2.5.4 Uniqueness of the stress-energy tensor

It's important to notice that the definition of the stress-energy tensor is not unique. If we add to the tensor a derivative of the form

$$T^{\mu\nu} + \partial_\lambda K^{\mu\lambda\nu}, \quad \text{with} \quad K^{\mu\lambda\nu} = -K^{\lambda\mu\nu}, \tag{2.97}$$

(i.e., where K is anti-symmetric in its first two indices), then the continuity condition $\partial_\mu T^{\mu\nu} = 0$ (2.88) will remain unchanged. To see this, note:

$$\partial_\mu \partial_\lambda K^{\mu\lambda\nu} = -\partial_\mu \partial_\lambda K^{\lambda\mu\nu} = -\partial_\lambda \partial_\mu K^{\lambda\mu\nu} = -\partial_\mu \partial_\lambda K^{\mu\lambda\nu} = 0,$$

where we used the anti-symmetry of K, then the commutativity of the derivative, and then relabelled the dummy indices $\mu \leftrightarrow \lambda$. Since this derivative is zero, Eq. (2.88) will not be changed. It is also possible to show that the total energy and momentum (2.96) are not impacted by this change. We shall not go through the details here, but it turns out that the stress energy tensor may be uniquely chosen so that it is symmetric (see, e.g., Landau Sec. 84[9]).

[9] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (1971).

2.6 Electromagnetic interactions

You are likely aware that electric and magnetic fields are described by (three-) vector, rather than scalar, quantities. As it turns out, electromagnetic fields and their interactions may be described by a (four-) vector field, denoted A^μ, which is called the *electromagnetic vector potential*. We will start by considering the motion of a particle within the influence of such a vector field. At first, we will consider again the weakly interacting case, where the influence of the particle on the field may be neglected. Therefore, on top of the particle terms, we need only consider interactions terms in the action. (Of course, the full theory will describe the evolution of the fields, which will be impacted by the particles; we will return to this in the next section.)

To determine how we should change the action of the particle to account for its interaction with the field, we can ask what scalars can be made from a vector field, A^μ? The simplest such term we could add to the action is proportional to

$$A_\mu \mathrm{d}x^\mu.$$

This is of course not the only option; we could also have any term of the form $f(A_\mu)\mathrm{d}s$, where f is a general scalar function. Ultimately, we must be guided by experiment. It turns out that the simplest term above is sufficient, as we shall see; we will return to a discussion of further possibilities in time.

The contribution to the action is therefore written

$$S_{\text{int}} = -\frac{q}{c}\int A_\mu \mathrm{d}x^\mu = -q\int \left(A_0 - \frac{1}{c}\boldsymbol{A}\cdot\boldsymbol{v}\right)\mathrm{d}t, \tag{2.98}$$

where q is an arbitrary scalar constant (negative sign and the factor of c are introduced for dimensional convenience), and x refers to the particle coordinates. To be dimensionally correct, the combination qA_μ must have energy units. The constant q is called the *charge* of the particle. Notice that as $q \to 0$, the interaction term will have no impact on the equations of motion; we say that the charge quantifies the strength of the interaction between the particle and the field. We note here that we employ the *Heaviside-Lorentz* units for electromagnetic quantities, in which the factors ε_0 and μ_0 do not appear explicitly in the equations; see Appendix 2.9.1 for discussion and definitions. For this section, we will retain factors of c, for ease of comparison to non-relativistic results.

Combining with Eq. (2.39), we may write the total Lagrangian as

$$L = -mc^2/\gamma - qA_0 + \frac{q}{c}\boldsymbol{A}\cdot\boldsymbol{v}, \tag{2.99}$$

where we have $A^\mu = (A_0, \boldsymbol{A})$; A_0 may be called the *electric scalar potential* (often denoted Φ), and $\boldsymbol{A}$ the *magnetic vector potential*; we shall see why shortly.

Directly from the Lagrangian, we may find the expression for the generalised momentum

$$\boldsymbol{P} = \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v}\gamma + \frac{q}{c}\boldsymbol{A} = \boldsymbol{p} + \frac{q}{c}\boldsymbol{A}, \tag{2.100}$$

using $\boldsymbol{p} = m\boldsymbol{v}\gamma$ from Eq. (2.41).

2.6.1 Lorentz force law

We shall now find the equations of motion for the system. From Mechanics [Eq. (1.19)], the equations of motion can be expressed

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\partial L}{\partial \boldsymbol{x}}.$$

First, we find

$$\begin{aligned}\frac{\partial L}{\partial \boldsymbol{x}} &= -\frac{q}{c}\left[c\nabla A_0 - \nabla(\boldsymbol{A}\cdot\boldsymbol{v})\right] \\ &= -\frac{q}{c}\left[c\nabla A_0 - (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} - \boldsymbol{v}\times(\nabla\times\boldsymbol{A})\right],\end{aligned} \tag{2.101}$$

where we made use of the vector identity[10] (note that the spatial derivatives are performed at fixed velocity).

For $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}$, note that $\boldsymbol{A}$ is evaluated at the particle position $\boldsymbol{x}(t)$, and may also have an intrinsic time dependence. Therefore,

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \frac{\partial \boldsymbol{A}}{\partial t} + \frac{\partial \boldsymbol{A}}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A}, \tag{2.102}$$

and so the equations of motion are

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{q}{c}\left[c\nabla A_0 + \frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{v}\times(\nabla\times\boldsymbol{A})\right]. \tag{2.103}$$

Notice that the equations of motion depend only on *derivatives* of the potential A^μ. We are therefore led to define the fields

$$\boldsymbol{E} \equiv -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla A_0 \quad \text{and} \quad \boldsymbol{B} \equiv \nabla\times\boldsymbol{A}, \tag{2.104}$$

where $\boldsymbol{E}$ is called the *electric field strength*, and $\boldsymbol{B}$ the *magnetic field strength*. With these definitions, the equations of motion may be written as

$$\boldsymbol{F} = q\boldsymbol{E} + \frac{q}{c}\boldsymbol{v}\times\boldsymbol{B}, \tag{2.105}$$

which is known as the *Lorentz force law*.

[10] $\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times(\nabla\times\boldsymbol{b}) + \boldsymbol{a}\times(\nabla\times\boldsymbol{a})$; see Appendix 2.9.2.

The equations for the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields (2.104) are linked through the potential, A_μ. It is also possible to form a set of equations for the fields, independently from the potential. Taking the curl of $\boldsymbol{E}$, and the divergence of $\boldsymbol{B}$, we find

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \quad \text{and} \quad \nabla \cdot \boldsymbol{B} = 0, \tag{2.106}$$

where we made use of the fact that the curl of a divergence, and the divergence of a curl, is zero (see Appendix 2.9.2). You may recognise these as the homogeneous pair of Maxwell's equations (Maxwell–Faraday equation, and Gauss's law for magnetism), which follow directly from the definition of the fields. The full equations governing the dynamics of the electromagnetic fields (i.e., the second inhomogeneous pair of Maxwell's equations) will require a consideration of the action for the fields themselves; we will return to this question in the coming sections.

***Problem* 2.7:** Show directly that $(\boldsymbol{v} \cdot \nabla)\boldsymbol{A} + v \times (\nabla \times \boldsymbol{A}) = \nabla(\boldsymbol{A} \cdot \boldsymbol{v})$, where $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{x})$, and $\boldsymbol{v}$ is a constant.

2.6.2 Gauge invariance

It's important to notice that the equations of motion (the Lorentz force law) depend only on the electromagnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, not on the potentials themselves. Since the fields (2.104) depend on *derivatives* of the potential, it is clear that the equations of motion are invariant under the global transformation

$$A_\mu \to A'_\mu = A_\mu + \text{const.}$$

In fact, it turns out the symmetry is much deeper, and the equations of motion remain invariant under the *local* transformation:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \theta(x^\mu), \tag{2.107}$$

where $\theta = \theta(x^\mu)$ is *any* (continuous, differentiable) function of spacetime coordinates. In terms of the scalar and (three) vector potentials, this is

$$A_0 \to A_0 + \frac{1}{c}\frac{\partial \theta}{\partial t} \quad \text{and} \quad \boldsymbol{A} \to \boldsymbol{A} - \nabla\theta.$$

The symmetry under this type of transformation is known as *gauge invariance*.

It's fairly straight forward to show that the fields in Eq. (2.104) remain invariant under this transformation. It is possible to show the equations obey this symmetry at a deeper level. The change to the term in the action (2.98) under the gauge transformation (2.107) is

$$A_\mu \mathrm{d}x^\mu \to A_\mu \mathrm{d}x^\mu + (\partial_\mu \theta)\mathrm{d}x^\mu \tag{2.108}$$

which is a total differential, and thus changes the action by at most a constant. It therefore cannot impact the equations of motion. That the electromagnetic interactions are observed to obey this gauge symmetry is the reason the $\int A^\mu A^\mu \mathrm{d}s$ term, which is not gauge invariant, may not appear in the action.

While any gauge choice is valid, some choices may be more convenient for certain problems. A common example is the *Lorenz* gauge, for which

$$\partial_\mu A^\mu = 0. \tag{2.109}$$

This choice is widely used, and is convenient due to the explicit Lorentz invariance. The Lorenz condition is known as a *partial* gauge condition, since it does not completely determine the gauge. Any further transform of the form $A^\mu \to A^\mu + \partial^\mu \phi$ where $\partial_\mu \partial^\mu \phi = 0$ will change the gauge while leaving the Lorenz condition intact.

A particularly common choice is the *Coulomb gauge*, also known as the transverse gauge, which is explicitly not Lorentz invariant. In the Coulomb gauge we assert that the spatial derivatives are zero:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0. \tag{2.110}$$

This choice is particularly common in practical and non-relativistic calculations. In general, it is always possible to choose a gauge where any one component of A_μ to be zero; it is often convenient to set $A_0 = 0$.

2.6.3 Electromagnetic field tensor

In the above sections, we derived the Lorentz force law, the equations of motion of a charged particle in an electromagnetic field. Despite beginning from a Lorentz invariant action, we resulted in an equation in terms of regular three-vectors. Of course, it should be possible to write the equations in four-vector (i.e., covariant) form.

Consider the action for a particle in an electromagnetic field:

$$S = \int \left(-m\,\mathrm{d}s - qA_\mu \mathrm{d}x^\mu\right),$$

(where we maintain the weak interaction approximation as discussed above, and go back to $c = 1$ units). Noting that $\mathrm{d}s = \sqrt{\mathrm{d}x_\mu \mathrm{d}x^\mu}$, we can write the variation in $\mathrm{d}s$ as

$$\delta(\mathrm{d}s) = \frac{\mathrm{d}x_\mu}{\mathrm{d}s}\delta(\mathrm{d}x^\mu) = u_\mu \mathrm{d}(\delta x^\mu), \tag{2.111}$$

where u is the four-velocity. At the same time, we have

$$\delta(A_\mu \mathrm{d}x^\mu) = \delta(A_\mu)\mathrm{d}x^\mu + A_\mu \mathrm{d}(\delta x^\mu),$$

where we used $\delta(\mathrm{d}x^\mu) = \mathrm{d}(\delta x^\mu)$. Therefore, the variation in the action is

$$\delta S = \int \Big(- m u_\mu \mathrm{d}(\delta x^\mu) - q \left[\partial_\nu A_\mu \delta x^\nu \mathrm{d}x^\mu + A_\mu \mathrm{d}(\delta x^\mu) \right] \Big), \tag{2.112}$$

where we made use of

$$\delta(A_\mu) = \partial_\nu A_\mu \delta(x^\nu).$$

Integrating the first and third terms in (2.112) by parts, this becomes

$$\begin{aligned} \delta S &= \int \Big(m \, \mathrm{d}u_\mu \, \delta x^\mu - q \left[\partial_\nu A_\mu \delta x^\nu \mathrm{d}x^\mu - \partial_\nu A_\mu \mathrm{d}x^\nu \, \delta x^\mu \right] \Big) \\ &= \int \Big(m \, \mathrm{d}u_\mu - q \left[\partial_\mu A_\nu \mathrm{d}x^\nu - \partial_\nu A_\mu \mathrm{d}x^\nu \right] \Big) \delta x^\mu = 0, \end{aligned} \tag{2.113}$$

where we swapped the dummy indices $\mu \leftrightarrow \nu$ in the second term. Finally, noting that the variation must be zero for arbitrary δx^μ, the equations of motion are found

$$m \, \mathrm{d}u_\mu - q \left[\partial_\mu A_\nu - \partial_\nu A_\mu \right] \mathrm{d}x^\nu = 0. \tag{2.114}$$

Just as above, we find the equations of motion depend only on the derivatives of the A field. We thus now introduce the *Electromagnetic field tensor* (sometimes called the Faraday tensor, or the Maxwell tensor), defined

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{2.115}$$

which is an anti-symmetric, rank two tensor. From the definitions in Eq. (2.104), the components can be found to be[11]

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{2.116}$$

It is important to notice that, with this definition, Lorentz transformations can be seen to directly mix up the components of electric and magnetic fields.

With this, the Lorentz force law (2.114) may be written in covariant form:

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}s} = \frac{q}{c^2} F^{\mu\nu} u_\nu. \tag{2.117}$$

It can be convenient to notice that the components of the electromagnetic fields can be expressed as

$$E^i = -F^{0i}, \quad \text{and} \quad B^i = -\frac{1}{2} \varepsilon^{ijk} F_{jk} \quad \text{or} \quad F^{ij} = -\varepsilon^{ijk} B^k = \varepsilon^{ijk} B_k, \tag{2.118}$$

[11] Note the potentially confusing notation: $a_x = a^1$ is the x-component of $\boldsymbol{a}$.

where ε^{ijk} is the totally anti-symmetric Levi-Civita symbol. Landau[12] introduces the compact notation $F^{\mu\nu} \equiv (-\boldsymbol{E}, \boldsymbol{B})$, with which $F_{\mu\nu} = (\boldsymbol{E}, \boldsymbol{B})$.

We may now express the homogeneous pair of Maxwell's equations (2.106) in covariant form. To do this, consider derivatives of the tensor terms of the form

$$\partial_\lambda F_{\mu\nu} = \partial_\lambda(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_\mu\partial_\lambda A_\nu - \partial_\nu\partial_\lambda A_\mu, \tag{2.119}$$

where we commuted the derivatives. Noting that we may write $\partial_\lambda A_\nu = F_{\lambda\mu} + \partial_\mu A_\lambda$, we see that we can form combinations that sum to zero:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} \equiv \partial_{[\lambda} F_{\mu\nu]} = 0. \tag{2.120}$$

This is called the *Bianchi identity*. Evaluating the Bianchi identity with $\{\lambda, \mu, \nu\} = \{0, 1, 2\}$ and $\{1, 2, 3\}$, respectively, we eventually arrive at

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \text{and} \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{2.121}$$

which are of course the Maxwell–Faraday equation and Gauss's law for magnetism. Therefore, Eq. (2.120) is the covariant form for the homogeneous pair of Maxwell's equations.

It is convenient to also define the *dual tensor* $\widetilde{F}^{\mu\nu}$ (sometimes denoted $\mathcal{F}$):

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{2.122}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally anti-symmetric rank four tensor.[13] Using the Landau notation, $\widetilde{F}^{\mu\nu} = (-\boldsymbol{B}, -\boldsymbol{E})$. With the dual tensor, the homogeneous Maxwell equations can be expressed as

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0. \tag{2.123}$$

***Problem* 2.8:** Explicitly determine the components $F_{\mu\nu}$ of the Faraday tensor using its definition in Eq. (2.115) and the definition of the fields (2.104).

***Problem* 2.9:** Show explicitly that the Lorentz force law (2.105) follows directly from the spatial terms in Eq. (2.117).

***Problem* 2.10:** What is the physical meaning of the temporal term in Eq. (2.117)?
Answer (2.10): It is the work equation: $\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = q\boldsymbol{E} \cdot \boldsymbol{v}$.

***Problem* 2.11:** Show explicitly from its definition (2.115), that the field tensor $F^{\mu\nu}$ is invariant under the gauge transformation (2.107).

***Problem* 2.12:** Find the scalar $F_{\mu\nu}F^{\mu\nu}$.
Solution (2.12): With notation $F = F^{\mu\nu}$, and noting that $F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta} = \eta F \eta$, we have $F_{\mu\nu}F^{\mu\nu} = \mathrm{tr}[(\eta F \eta)^{\mathrm{T}} F] = -\mathrm{tr}[(\eta F \eta) F] = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2)$, where $\mathrm{tr}(M) = M^{\mu\mu}$ is the trace.

***Problem* 2.13:** Explicitly work out equations (2.121) from the Bianchi identity (2.120), and from the dual tensor equation (2.123).

[12] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (1971).

[13] $\epsilon^{\mu\nu\rho\sigma} = 1$ under any even permutation of ϵ^{0123}, $= -1$ under any odd permutation, $= 0$ if any two indices are equal, and $\epsilon^{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}$.

2.6.4 Current density

Before we continue to full field theory description of electromagnetic fields, we require a way to deal properly with the concept of particle point charges in the context of a continuous field theory. We can define the *charge density*, ρ, for a collection of point particles as

$$\rho \equiv \sum_i q_i \delta(\boldsymbol{x} - \boldsymbol{x}_i), \tag{2.124}$$

where $\delta(\boldsymbol{x}) = \delta(x_1)\delta(x_2)\delta(x_3)$ is the Dirac delta function, defined via

$$\int_{x_0-\delta}^{x_0+\delta} f(x)\delta(x - x_0)\,\mathrm{d}x = f(x_0).$$

The total charge in a volume, $Q \equiv \sum_i q_i$, is clearly the volume integral of the charge density:

$$Q = \int \mathrm{d}V\,\rho = \sum_i q_i. \tag{2.125}$$

This is not surprising, of course, but shows our definitions are consistent.

From the above, we may identify $\mathrm{d}q \equiv \rho\,\mathrm{d}V$. Note that q (and thus Q) is a scalar by its definition [see Eq. (2.98)], and therefore so is dq. On the other hand, ρ is not a scalar; only the combination $\rho \mathrm{d}V$ is. If we multiply by the vector $\mathrm{d}x^\mu$, we have

$$\mathrm{d}q\,\mathrm{d}x^\mu = \rho\,\mathrm{d}^4x\,\frac{\mathrm{d}x^\mu}{c\,\mathrm{d}t}. \tag{2.126}$$

Note that $\mathrm{d}q\mathrm{d}x^\mu$ is a four vector, and $\mathrm{d}^4x = c\mathrm{d}t\,\mathrm{d}V$ is a scalar. Therefore $\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}t}$ is a four vector, so we define the *current density* four vector

$$j^\mu \equiv \rho\frac{\mathrm{d}x^\mu}{\mathrm{d}t}, \tag{2.127}$$

which has components $j^\mu = (j^0, \boldsymbol{j})$. The definition as a current density is justified: j^0 is (up to a constant of c) simply the charge density, $j^0 = c\rho$, and the spatial terms are $\boldsymbol{j} = \rho\boldsymbol{v}$. It is sometimes convenient to factor out the unit charge q from the definition of the current density, and define

$$\mathcal{J}^\mu = \frac{1}{q}j^\mu. \tag{2.128}$$

With these definitions, we can re-express the interaction contribution to the action (2.98) in terms of the current density by writing $q = \rho\,\mathrm{d}V$:

$$\begin{aligned} S_{\text{int.}} &= -\frac{1}{c}\int \rho\,\mathrm{d}V A_\mu \mathrm{d}x^\mu = -\frac{1}{c}\int \rho\frac{\mathrm{d}x^\mu}{\mathrm{d}t}A_\mu\,\mathrm{d}V\mathrm{d}t \\ &= -\frac{1}{c^2}\int j^\mu A_\mu\,\mathrm{d}^4x. \end{aligned} \tag{2.129}$$

We therefore identify the interaction contribution to the Lagrangian density as:

$$\mathcal{L}_{\text{int.}} = -\frac{1}{c} j^\mu A_\mu, \quad \text{or} \quad \mathcal{L}_{\text{int.}} = -\frac{q}{c} \mathcal{J}^\mu A_\mu, \tag{2.130}$$

which may be used for general charge distributions.

We shall now consider the law of the conservation of charge, by examining the four-divergence of the current density, $\partial_\mu j^\mu$. Integrating this divergence over some finite volume, V, we have

$$\begin{aligned} \int \partial_\mu j^\mu \, \mathrm{d}V &= \frac{\partial}{\partial t} Q_V + \int \nabla \cdot (\rho \boldsymbol{v}) \, \mathrm{d}V \\ &= \frac{\partial}{\partial t} Q_V + \oint_S \rho \boldsymbol{v} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S, \end{aligned} \tag{2.131}$$

where Q_V is the total charge within the volume, and we used Gauss' theorem for the second term [see Appendix 2.9.2], with S the closed surface bounding the volume and $\hat{\boldsymbol{n}}$ the unit vector normal to the surface. The first term on the right-hand-side of Eq. (2.131) is the rate of change of the total charge contained in the volume. The second term is the rate at which charge passes through the surface of the volume (see Sec. 2.5.1). If we assert that the charge must be a locally conserved quantity, then the these terms must sum to zero, leading to

$$\partial_\mu j^\mu = 0, \tag{2.132}$$

which is called the *continuity* equation. While we have asserted this fact based on physical arguments, we shall later see that this law is closely tied to the principle of gauge invariance.

***Problem* 2.14:** In the rest frame of a charge distribution (K'), we have $J^\mu = (\rho_0/c, \mathbf{0})$. Observed from another frame (K), the charge is observed to move along the x-axis with velocity v. (i) What is the current as observed in the K frame? (ii) As we saw above, the total charge $Q = \int \mathrm{d}V j^0$ is a scalar, while the density, $\rho = j^0/c$ is not. Write the current in terms of the density ρ in the current frame.
Answer (2.14): i: $(c\rho_0\gamma, v\rho_0\gamma, 0, 0)$, ii: $(c\rho, v\rho, 0, 0)$

2.7 Electrodynamics: Maxwell's equations

We now seek the set of equations that govern the evolution of the A_μ fields themselves. To construct the required action, noting the principles we discussed above, we may consider scalar functions of the fields A_μ and their first derivatives $\partial_\nu A_\mu$. Further, since from experience, we expect the the electromagnetic fields to obey the superposition principle, we restrict our consideration to terms that are quadratic in the A_μ fields. Finally, since the equations of motion we found before were gauge invariant, we shall consider only terms which are themselves gauge invariant.

The terms proportional to $A_\mu A^\mu$ or $\partial_\mu A^\mu$ are not gauge invariant. Under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \theta$ [Eq. (2.107)], they become

$$\begin{aligned} &A_\mu A^\mu \to A_\mu A^\mu + 2A_\mu \partial^\mu \theta + \partial_\mu \theta \partial^\mu \theta, \quad \text{and} \\ &\partial_\mu A_\nu \to \partial_\mu A_\nu + \partial_\mu \partial_\nu \theta. \end{aligned} \tag{2.133}$$

The anti-symmetric combination $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (2.115), however, can be written purely in terms of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, which are gauge invariant. Therefore, the scalar $F^{\mu\nu}F_{\mu\nu}$ (often written as F^2 for brevity) is the only scalar quantity that is both gauge invariant, and second-order in the field.

Therefore, the term in the Lagrangian involving only the fields may be

$$\mathcal{L}_\mathrm{F} = \frac{-1}{4} F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2). \tag{2.134}$$

The factor $1/4$ is arbitrary, and corresponds to a choice of units. Combining with the interaction term from Eq. (2.130), the total Lagrangian including all field terms is

$$\mathcal{L} = -\frac{1}{c} j^\mu A_\mu - \frac{1}{4} F^{\mu\nu}F_{\mu\nu}. \tag{2.135}$$

This is often termed the *Maxwell Lagrangian*.

To determine the field equations, we apply the Euler-Lagrange condition (2.54) with $\phi = A_\nu$. We have (see Problem 2.15)

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}, \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = -\frac{1}{c} j^\nu, \tag{2.136}$$

so the field equations are

$$\partial_\mu F^{\mu\nu} = \frac{1}{c} j^\nu. \tag{2.137}$$

To write these in three-dimensional form, we evaluate (2.137) with $\nu = 0$ and $\nu = \{1, 2, 3\}$, respectively, resulting in:

$$\nabla \cdot \boldsymbol{E} = \rho, \quad \text{and} \qquad \nabla \times \boldsymbol{B} = \frac{1}{c}\left(\boldsymbol{j} + \frac{\partial \boldsymbol{E}}{\partial t}\right). \tag{2.138}$$

These are of course Gauss's law and Ampère's law; we thus recognise Eq. (2.137) as the inhomogeneous Maxwell equations. To summarise, in the covariant notation, the Lorentz force law and Maxwell's equations can be summarised with

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}s} = \frac{q}{c^2} F^{\mu\nu} u_\nu, \qquad \partial_\mu \widetilde{F}^{\mu\nu} = 0, \quad \text{and} \quad \partial_\mu F^{\mu\nu} = \frac{1}{c} j^\nu. \tag{2.139}$$

***Problem* 2.15:** Derive the derivative $\frac{\partial F^2}{\partial(\partial_\mu A_\nu)} = 4F^{\mu\nu}$, required in Eq. (2.136).

Solution (2.15): For brevity, write $\partial_\mu A_\nu \equiv t_{\mu\nu}$. Then,

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= \eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}F_{\mu\nu} = \eta^{\alpha\mu}\eta^{\beta\nu}(t_{\alpha\beta}-t_{\beta\alpha})(t_{\mu\nu}-t_{\nu\mu}) \\ &= \eta^{\alpha\mu}\eta^{\beta\nu}\left(2t_{\alpha\beta}t_{\mu\nu}-2t_{\alpha\beta}t_{\nu\mu}\right), \end{aligned}$$

where we used the symmetry of the metric $\eta^{\mu\nu}=\eta^{\nu\mu}$, and the fact that we may relabel the dummy indices ($\mu\leftrightarrow\nu$, $\alpha\leftrightarrow\beta$). Noting that $\frac{\partial t_{\mu\nu}}{\partial t_{\sigma\rho}} = \delta^\sigma_\mu\delta^\rho_\nu$ is zero unless $\mu=\sigma$ and $\nu=\rho$,

$$\frac{\partial}{\partial t_{\sigma\rho}}\left(\eta^{\alpha\mu}\eta^{\beta\nu}t_{\alpha\beta}t_{\mu\nu}\right) = \eta^{\sigma\mu}\eta^{\rho\nu}t_{\mu\nu}+\eta^{\alpha\sigma}\eta^{\beta\rho}t_{\alpha\beta} = 2t^{\sigma\rho}.$$

Therefore

$$\frac{\partial}{\partial t_{\sigma\rho}}\left(F^{\mu\nu}F_{\mu\nu}\right) = 4\left(t^{\sigma\rho}-t^{\rho\sigma}\right) \quad\Longrightarrow\quad \frac{\partial}{\partial(\partial_\sigma A_\rho)}\left(F^{\mu\nu}F_{\mu\nu}\right) = 4F^{\sigma\rho}.$$

2.7.1 Gauge invariance and charge conservation

The gauge invariance and the charge conservation in electrodynamics are intrinsically linked. In the full quantum field theory, it turns out that the conserved Noether current (2.77) associated with the gauge invariance (2.107) is just the electric charge current (2.127). Without a concept of the gauge symmetry for the matter part of the problem (i.e., the point charges or charge density), this doesn't quite work. Even without the full quantum theory, however, we can still show that these concepts are closely connected.

Under the gauge transformation Eq. (2.107), the variation of the Maxwell Lagrangian (2.135) is

$$\delta\mathcal{L} = \frac{-1}{c}j^\mu\partial_\mu\theta, \tag{2.140}$$

which follows simply due to the explicit gauge invariance of the F^2 term (2.134). The corresponding contribution to the action can be written as

$$\delta S = \frac{-1}{c^2}\int j^\mu\,\partial_\mu\theta\,\mathrm{d}^4x = \frac{1}{c^2}\int\left(\partial_\mu j^\mu\right)\theta\,\mathrm{d}^4x, \tag{2.141}$$

where we used integration by parts, and assumed the current terms go to zero at infinity. Since, by assertion, the gauge function θ may be *any* function of time and coordinates, it follows that this term is zero only if $\partial_\mu j^\mu = 0$. Therefore, the gauge invariance implies the continuity equation (2.132), which, as discussed above, is the condition for local charge conservation.

2.7.2 Energy of the electromagnetic field

From Maxwell's equations (2.121) and (2.138), we can form the expressions

$$\boldsymbol{E}\cdot\nabla\times\boldsymbol{B} = \frac{1}{c}\boldsymbol{E}\cdot\left(\frac{\partial\boldsymbol{E}}{\partial t}+\boldsymbol{j}\right), \quad\text{and}\quad \boldsymbol{B}\cdot\nabla\times\boldsymbol{E} = -\frac{1}{c}\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t}. \tag{2.142}$$

Subtracting these, and making use of the vector identity[14], we have

$$\frac{1}{2}\frac{\partial}{\partial t}\left(E^2 + B^2\right) = -\boldsymbol{E}\cdot\boldsymbol{j} - \nabla\cdot[c\boldsymbol{E}\times\boldsymbol{B}]. \tag{2.143}$$

It is customary to define the term in brackets as the *Poynting vector*:

$$\boldsymbol{S} \equiv c\boldsymbol{E}\times\boldsymbol{B}. \tag{2.144}$$

To elucidate the physical meaning of this equation (2.143), we shall integrate it over a volume, V. Recognising the current density as being that from a collection of point particles [Eq. (2.124)], $\boldsymbol{j}(x) = \sum q_i \boldsymbol{v}_i \delta(x - x_i)$, we have

$$\frac{\partial}{\partial t}\int\left(\frac{E^2+B^2}{2}\right)\mathrm{d}V + \sum_i q_i \boldsymbol{E}\cdot\boldsymbol{v}_i = -\int \nabla\cdot\boldsymbol{S}\,\mathrm{d}V. \tag{2.145}$$

From Problem 2.10, the rate of change in kinetic energy of a particle of charge q in an electromagnetic field is given

$$\frac{\mathrm{d}\mathcal{E}_{\text{part.}}}{\mathrm{d}t} = q\boldsymbol{E}\cdot\boldsymbol{v}.$$

Therefore, we may express Eq. (2.145) as

$$\frac{\partial}{\partial t}\left(\int \mathcal{U}\,\mathrm{d}V + \sum \mathcal{E}_{\text{part.}}\right) = -\oint \boldsymbol{S}\cdot\mathrm{d}\boldsymbol{\Omega}, \tag{2.146}$$

where we used Gauss' theorem [Eq. (2.167) in the appendix] for the term on the right-hand-side, with $\mathrm{d}\boldsymbol{\Omega}$ being the surface element. If the volume is taken to include all space, and the fields are assumed to go to zero at infinity, then the surface integral is zero, and it's clear that

$$\mathcal{U} \equiv \frac{E^2+B^2}{2} \tag{2.147}$$

can be interpreted as the *energy density* of the electromagnetic fields, assuring the energy conservation. If the volume is taken to be finite, then the term on the right-hand-side is non-zero. In this case, it clear that the Poynting vector, $\boldsymbol{S}$, should be interpreted as the energy density flux (energy transfer per unit area, per unit time).

2.7.3 Electromagnetic stress-energy tensor

We will now calculate the *Maxwell stress-energy tensor*, in the source free ($j^\mu = 0$) case. From Eq. (2.87),

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L},$$

[14] $\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot\nabla\times\boldsymbol{a} - \boldsymbol{a}\cdot\nabla\times\boldsymbol{b}$; see Appendix 2.9.2.

where the Maxwell Lagrangian is given by Eq. (2.135). Using the derivatives we already computed (2.136), with the field being the components of the vector potential (i.e., $\phi = A_\lambda$) the expression is readily found

$$T^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}\eta^{\mu\nu}F^{\rho\lambda}F_{\rho\lambda}. \tag{2.148}$$

Note, however, that this expression is not symmetric, and is also not gauge invariant.

It's clear that we could form a symmetric tensor by adding to Eq. (2.148) a term of the form

$$F^{\mu\lambda}\partial_\lambda A^\nu. \tag{2.149}$$

But is this justified? Note that, in the absence of charges, the equations of motion read $\partial_\lambda F^{\mu\lambda} = 0$. Therefore, we can write:

$$F^{\mu\lambda}\partial_\lambda A^\nu = \partial_\lambda(F^{\mu\lambda}A^\nu). \tag{2.150}$$

Since this term is of the form of Eq. (2.97), we may safely add it to the $T^{\mu\nu}$ tensor (i.e., it does not break the condition $\partial_\mu T^{\mu\nu} = 0$). With this, we arrive at the symmetric stress-energy tensor:

$$T^{\mu\nu} = F^{\mu\lambda}F_\lambda{}^\nu + \frac{1}{4}\eta^{\mu\nu}F^{\rho\lambda}F_{\rho\lambda}. \tag{2.151}$$

From the relations in Sec. 2.6.3, the temporal component, which should correspond to the energy density, is readily found to be

$$T^{00} = \boldsymbol{E}^2 + \frac{1}{4}F^2 = \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right), \tag{2.152}$$

exactly as expected from classical electromagnetism, Eq. (2.147).

As a final note, we mention that we could have essentially *guessed* the form of the stress-energy tensor from the knowledge that the energy density is second-order in the electric and magnetic fields. There are only two tensors we can form from the F (or A) fields that are gauge independent, symmetric, and second-order in the fields:

$$T^{\mu\nu} = a\,F^{\mu\lambda}F_\lambda{}^\nu + b\,\eta^{\mu\nu}F^2.$$

The constants a and b may then be relatively easily found from the requirement that T^{00} equals the well-known energy density (2.147).

2.8 Electromagnetic waves and radiation

One important thing to notice about Maxwell's equations is that there are non-trivial solutions, even in the case where there are no charges. Such solutions are known as free (or vacuum) *electromagnetic waves*.

In the absence of charges, we have

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 0 \\ \implies \partial^2 A^\nu - \partial^\nu \partial_\mu A^\mu &= 0. \end{aligned} \tag{2.153}$$

As we discussed above, we are free to choose the Lorenz gauge Eq. (2.109), where $\partial_\mu A^\mu = 0$. In this case, the above equation becomes (multiplying by c^2):

$$c^2 \partial^2 A^\nu = \left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2 \right) A^\mu = 0, \tag{2.154}$$

which you may recognise as the *wave equation* (equivalent to the Klein-Gordon equation (2.66) with $\mu = 0$). It can be seen that functions of the form

$$A^\mu(x_\mu) = \epsilon^\mu a_0 e^{\pm i k_\mu x^\mu} \tag{2.155}$$

are solutions, where ϵ^μ is a unit four-vector termed the *polarisation vector*, a_0 is an arbitrary coefficient (called the *amplitude*), k_μ is an arbitrary four vector, and it is understood that the real part of the solution is taken. If we write the components of this vector as $k^\mu = (\omega/c, \boldsymbol{k})$, then these solutions may be identified as plane waves of angular frequency ω, and wavevector $\boldsymbol{k}$ (which are related to the frequency $f = \omega/(2\pi)$ and wavelength $\lambda = 2\pi/|\boldsymbol{k}|$). The solution describes a plane wave travelling along the direction $\boldsymbol{k}$, with speed c.

Of course, general solutions may be much more complicated. However, since any linear combination of plane wave solutions (2.155) is also a solution, general solutions may be formed from linear combinations of plane wave solutions with different amplitudes and k^μ vectors (Fourier theorem). Solutions that can be written in the form of Eq. (2.155) – i.e., that involve only a single frequency ω – are said to be *monochromatic*.

From the wave equation (2.154), we find the condition on k_μ:

$$k^\mu k_\mu = 0. \tag{2.156}$$

Combining this with the definition of the components of k_μ, we see $\omega^2 = c^2 \boldsymbol{k}^2$, and

$$\boldsymbol{k} = \frac{\omega}{c} \hat{\boldsymbol{n}}, \tag{2.157}$$

where $\hat{\boldsymbol{n}}$ is the unit vector along the direction of propagation of the wave. Further, taking the derivative of the solution (2.155), and noting the Lorenz gauge condition $\partial_\mu A^\mu = 0$, we find

$$k_\mu \epsilon^\mu = 0, \tag{2.158}$$

so the polarisation must be perpendicular to the direction of propagation.

2.9 Appendix

2.9.1 Unit systems for electrodynamics

In defining the units for electrodynamics, one must make choices in linking the definitions of charges and field strengths to the usual mechanical units of length L, mass M, and time T. For example, in Coulomb's law:

$$F = k_e \frac{q_1 q_2}{r^2},$$

the value taken for k_e depends on the definition of charges in terms of the usual mechanical units (and similarly for magnetic fields). In SI units, for example, charges are measured in Coulombs, and k_e is taken to be $1/(4\pi\varepsilon_0)$. It's important to keep track of the unit system being used; in electrodynamics, not only do the values of constants change, but the form of the equations change also.

Within the SI system of units, the vacuum permittivity ε_0, and vacuum permeability μ_0, are linked via[15]

$$\varepsilon_0\mu_0 = 1/c^2, \qquad \text{and} \qquad \mu_0 = \frac{4\pi\alpha\hbar}{e^2 c}, \qquad \text{or} \qquad 4\pi\varepsilon_0 = \frac{e^2}{\hbar c\,\alpha}. \tag{2.159}$$

The SI system is a so-called *rationalised* unit system, where no 4π factors appear in Maxwell's equations; instead they appear in the inverse-square force laws (i.e., Coulomb's law and the Biot-Savart law). It is also possible to choose a *non-rationalised* unit system, where the 4π factors do not appear in the inverse-square force laws, but instead in Maxwell's equations.

Particularly in theoretical physics, it is extremely common to work with units systems where ε_0 and μ_0 do not appear in the equations (this can be thought of as by defining units in which $\varepsilon_0 = \mu_0 = 1$, or by absorbing them into the definition of the charges/fields). Two common choices are the *Heaviside-Lorentz* system, which is a rationalised system, and the *Gaussian* system, which is non-rationalised. The Gaussian system was very common, particularly in older textbooks. The Heaviside-Lorentz system (rationalised) is widely used in relativistic and particle physics. In the Gaussian system, the Coulomb law constant is taken to be $k_e = 1$, meaning charges have dimension $\sqrt{M.L^3}/T$. The Heaviside-Lorentz unit system is similar, except that it is rationalised; the constant is taken to be $k_e = 1/(4\pi)$. In both the Heaviside-Lorentz and Gaussian systems, the electric and magnetic field have the same units as each other; in the SI system, they differ by the velocity units L/T.

[15] Of these constants, only the fine structure constant, $\alpha \approx 1/137.035\,999\,084(21)$ is a measured value. The elementary charge is defined *exactly* as $e \equiv 1.602\,176\,634 \times 10^{-19}$ C. Note that with this definition, $e = |e|$ and the electron has charge $-e$ (it is common to use the opposite convention). Similarly, the speed of light is defined exactly $c \equiv 299\,792\,458\,\mathrm{m\,s^{-1}}$, and Planck's constant as $h \equiv 2\pi\hbar = 6.626\,070\,15 \times 10^{-34}$ J s.

In the different systems, Coulomb's law takes the form:

$$F = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}}_{\text{SI}} = \underbrace{\frac{1}{4\pi}\frac{q_1 q_2}{r^2}}_{\text{HL}} = \underbrace{\frac{q_1 q_2}{r^2}}_{\text{Gaus.}}, \tag{2.160}$$

and the form of the Lorentz force law:

$$\boldsymbol{F} = \underbrace{q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})}_{\text{SI}} = \underbrace{q(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B})}_{\text{HL, Gaus.}}. \tag{2.161}$$

In all systems, $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, though

$$\boldsymbol{E} = \underbrace{-\frac{\partial\boldsymbol{A}}{\partial t} - \nabla A_0}_{\text{SI}} = \underbrace{-\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} - \nabla A_0}_{\text{HL, Gaus.}}. \tag{2.162}$$

The three-vector form of Maxwell's equations are:

SI:	Heaviside-Lorentz:	Gaussian:
$\nabla\cdot\boldsymbol{E} = \dfrac{\rho}{\varepsilon_0}$	$\nabla\cdot\boldsymbol{E} = \rho$	$\nabla\cdot\boldsymbol{E} = 4\pi\rho$
$\nabla\times\boldsymbol{B} = \mu_0(\boldsymbol{j} + \varepsilon_0\partial_t\boldsymbol{E})$	$\nabla\times\boldsymbol{B} = \dfrac{1}{c}(\boldsymbol{j} + \partial_t\boldsymbol{E})$	$\nabla\times\boldsymbol{B} = \dfrac{1}{c}(4\pi\boldsymbol{j} + \partial_t\boldsymbol{E})$
$\nabla\times\boldsymbol{E} = -\partial_t B$	$\nabla\times\boldsymbol{E} = -\dfrac{1}{c}\partial_t B$	$\nabla\times\boldsymbol{E} = -\dfrac{1}{c}\partial_t B$
$\nabla\cdot\boldsymbol{B} = 0$	$\nabla\cdot\boldsymbol{B} = 0$	$\nabla\cdot\boldsymbol{B} = 0.$

(2.163)

In the four-vector form:

$$\partial_\mu F^{\mu\nu} = \underbrace{\mu_0 j^\nu}_{\text{SI}} = \underbrace{\frac{1}{c}j^\nu}_{\text{HL}} = \underbrace{\frac{4\pi}{c}j^\nu}_{\text{Gaus.}}, \tag{2.164}$$

where, if $F^{\mu\nu} = (-\boldsymbol{E}, \boldsymbol{B})$ in the Heaviside-Lorentz and Gaussian systems, then $F^{\mu\nu} = (-\boldsymbol{E}/c, \boldsymbol{B})$ in the SI system. The Bianchi identity $\partial_{[\lambda}F_{\mu\nu]} = 0$, or

$$\partial_\mu\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = 0,$$

remains the same in all systems.

2.9.2 Some vector calculus identities

The fundamental theorem of calculus

$$\int_{x_1}^{x_2}\partial_x\phi(x)\,\mathrm{d}x = \phi(x_2) - \phi(x_1). \tag{2.165}$$

may be generalised to vector gradients

$$\int_{\boldsymbol{x_1}}^{\boldsymbol{x_2}} [\boldsymbol{\nabla}\phi(\boldsymbol{x})] \cdot \mathrm{d}\boldsymbol{l} = \phi(\boldsymbol{x_2}) - \phi(\boldsymbol{x_1}). \tag{2.166}$$

The generalisations for vector functions are known as the divergence theorem (or Gauss' theorem), and Stokes' theorem.

For a vector function $\boldsymbol{A}(\boldsymbol{x})$ within a volume V bounded by a closed surface S, the volume integral of the divergence of $\boldsymbol{A}$ is related to the surface integral of the component of $\boldsymbol{A}$ (outwardly) normal to the surface S as:

$$\int_V (\nabla \cdot \boldsymbol{A})\, \mathrm{d}V = \oint_S \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} \tag{2.167}$$

$\mathrm{d}V = \mathrm{d}^3x = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ is the volume element, and $\mathrm{d}\boldsymbol{S} = \hat{\boldsymbol{n}}\mathrm{d}S$ is the surface (area) element normal to the surface. This is known as the *divergence theorem*, or Gauss' theorem. This can be generalised to higher dimensions:

$$\int_\Omega \partial_\mu A^\mu \, \mathrm{d}\Omega = \oint_\omega A^\mu N_\mu \mathrm{d}\omega, \tag{2.168}$$

where Ω is the generalised (e.g., 4D) volume, ω is the closed surface bounding the volume, with generalised (e.g., 3D) surface element $N_\mu \mathrm{d}\omega$, where N_μ is the unit vector outwardly normal to the surface.

Similarly, for a vector function $\boldsymbol{A}$ within an arbitrary *open* surface S, the surface integral of the component of the curl of $\boldsymbol{A}$ normal to the surface is related to the line integral along L, the closed path which bounds the surface S as

$$\int_S (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} = \oint_L \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}, \tag{2.169}$$

where $\mathrm{d}\boldsymbol{l}$ is the line element of the path L.

Another useful identity is Green's theorem,

$$\begin{aligned} \int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) \mathrm{d}V &= \oint_S (\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi) \cdot \mathrm{d}\boldsymbol{S} \\ &= \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) \mathrm{d}|S|, \end{aligned} \tag{2.170}$$

which follows from Gauss' theorem, where $\mathrm{d}\boldsymbol{S} = \hat{\boldsymbol{n}}\mathrm{d}|S|$, and $\partial/\partial n$ is the (outward) normal derivative at the surface S.

A number of other identities are provided for convenience:

$$\begin{aligned} &\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}) \\ &[(\nabla \cdot \boldsymbol{a})\boldsymbol{b}] = (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} \\ &\nabla(\boldsymbol{a} \cdot \boldsymbol{b}) = (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} + \boldsymbol{a} \times (\nabla \times \boldsymbol{b}) + \boldsymbol{a} \times (\nabla \times \boldsymbol{a}) \\ &\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla \times \boldsymbol{a} - \boldsymbol{a} \cdot \nabla \times \boldsymbol{b} \\ &\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) \\ &\nabla \times (\nabla \times \boldsymbol{a}) = \nabla(\nabla \cdot \boldsymbol{a}) - (\nabla \cdot \nabla)\boldsymbol{a} \\ &\nabla \cdot (\nabla \times \boldsymbol{a}) = \nabla \times (\nabla\phi) = 0. \end{aligned} \tag{2.171}$$

With the index notation, and Einstein summation convention, we have

$$\nabla \cdot \boldsymbol{a} = \partial_i a^i \quad \text{and} \quad \boldsymbol{a} \times \boldsymbol{b} = \epsilon^{ijk} a_i b_j \tag{2.172}$$

where ϵ is the anti-symmetric Levi-Civita symbol (note that we have $\epsilon^{123} = +1$, and we avoid defining ϵ_{ijk}).

References

[1] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (1971).

[2] D. J. Griffiths, *Intoduction to Electrodynamics* (1999).

[3] L. Susskind and A. Friedman, *Special Relativity and Classical Field Theory* (2014).

[4] J. D. Jackson, *Classical Electrodynamics* (2001).

[5] A. Einstein, "Zur Elektrodynamik bewegter Körper", Annalen der Physik **322**, 891 (1905).

[6] *On the Electrodynamics of Moving Bodies (Translation)*, `https://www.fourmilab.ch/etexts/einstein/specrel/www/`.

[7] A. Einstein, "Ist Die Trägheit Eines Körpers von Seinem Energieinhalt Abhängig?", Annalen der Physik **323**, 639–641 (1905).

[8] *Does the Inertia of a Body Depend upon its Energy-Content? (Translation)*, `https://www.fourmilab.ch/etexts/einstein/E_mc2/www/`.

[9] E. Noether, "Invariante Variationsprobleme", Gott. Nachr. **1918**, 235–257 (1918).

[10] *Invariant Variation Problems (Translation)*, arXiv:physics/0503066.